

$\mathcal{N} = 2$ Conformal Superspace in Four Dimensions

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ABSTRACT: We develop the geometry of four dimensional $\mathcal{N} = 2$ superspace where the entire conformal algebra of $SU(2, 2|2)$ is realized linearly in the structure group rather than just the $SL(2, \mathbb{C}) \times U(2)_R$ subgroup of Lorentz and R -symmetries, extending to $\mathcal{N} = 2$ our prior result for $\mathcal{N} = 1$ superspace. This formulation explicitly lifts to superspace the existing methods of the $\mathcal{N} = 2$ superconformal tensor calculus; at the same time the geometry, when degauged to $SL(2, \mathbb{C}) \times U(2)_R$, reproduces the existing formulation of $\mathcal{N} = 2$ conformal supergravity constructed by Howe.

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1 Introduction

It has long been apparent that superconformal techniques have a key role to play in constructing supergravity theories.¹ For $\mathcal{N} = 1$ theories the role is quite well understood: any action involving supergravity coupled to matter can be described (often more easily!) by conformal supergravity coupled to some compensating field plus matter. Even in the absence of matter, quantizing supergravity is most easily done via the introduction of compensator fields, a topic which by now is textbook material [2, 3]. In the presence of matter, the compensator fields and conformal techniques are even more important [4].

The same can be said of $\mathcal{N} = 2$ supergravity theories, although they are not nearly so well understood as their $\mathcal{N} = 1$ cousins. A central tool in their analysis is the so-called superconformal tensor calculus, which for brevity's sake we will occasionally refer to as STC. This formalism, which allows the construction of component actions which respect superconformal invariance, was given for $\mathcal{N} = 2$ supergravity by de Wit, van Holten, and Van Proeyen [5], who along with various collaborators throughout the 1980s made extensive use of this technique [6–10]; for example, the various $\mathcal{N} = 2$ supergravity theories can be easily understood in the STC framework by coupling conformal supergravity to different compensator fields [8]. This method continues to be used to this day [11–13].

Superspace techniques also exist for $\mathcal{N} = 2$ supergravity in several flavors.² The approaches most closely related to the one we will take are that of Grimm [17], who chose $\text{SL}(2, \mathbb{C}) \times \text{SU}(2)_{\text{R}}$ for the structure group, and Howe [18], who extended this work to general \mathcal{N} with the structure group $\text{SL}(2, \mathbb{C}) \times \text{U}(\mathcal{N})_{\text{R}}$. These superspace formulations have recently been elaborated upon and used to describe supergravity in projective superspace [19, 20].³ Although such superspace techniques allow a certain super-Weyl transformation, the relation to the *manifestly* superconformal method of de Wit et al. has remained opaque, especially since the super-Weyl transformation is manifested inhomogeneously on the torsion superfields.

In this paper we will clarify this relation by constructing in superspace the *manifestly* superconformal geometry corresponding to $\mathcal{N} = 2$ conformal supergravity. That is, we will present a superspace, which we call $\mathcal{N} = 2$ conformal superspace, where the structure group is $\text{SU}(2, 2|2)$ and whose component form reduces to $\mathcal{N} = 2$ superconformal tensor calculus. Since Howe's formulation is well known to correspond to conformal supergravity with an unconstrained dilatation parameter [18, 20], it must (and, we will show, does) correspond to a certain gauge-fixed version of the theory we construct. These results extend to $\mathcal{N} = 2$ our previous work [21] where we constructed $\mathcal{N} = 1$ conformal superspace.

Prior experience with superspace might hint that a larger structure group would necessarily yield a more complicated theory. However, as we showed in [21], the Bianchi identities and curvature structure of the $\mathcal{N} = 1$ theory were actually *simpler* and super-

¹See for example the comprehensive review article by Fradkin and Tseytlin [1].

²See for example [14] and [15], as well as [16] for an early discussion of superconformal aspects of $\mathcal{N} = 2$ supergravity in superspace.

³It was shown in [20] that Grimm's formulation is a gauge-fixed version of Howe's.

ficially resembled super Yang-Mills.⁴ The same holds in $\mathcal{N} = 2$, where we will show that the constraint structure of $\mathcal{N} = 2$ conformal supergravity may be described by covariant derivatives with a simple algebra,

$$\begin{aligned}\{\nabla_\alpha{}^i, \nabla_\beta{}^j\} &= -2\epsilon^{ij}\epsilon_{\alpha\beta}\bar{\mathcal{W}}, & \{\bar{\nabla}^{\dot{\alpha}}{}_i, \bar{\nabla}^{\dot{\beta}}{}_j\} &= +2\epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\mathcal{W} \\ \{\nabla_\alpha{}^i, \bar{\nabla}_{\dot{\beta}j}\} &= -2i\delta_j^i\nabla_{\alpha\dot{\beta}}\end{aligned}$$

where \mathcal{W} is an $\mathcal{N} = 2$ field strength valued in the superconformal algebra and obeying certain constraints. As with our $\mathcal{N} = 1$ construction, the resemblance to super Yang-Mills significantly simplifies the algebra of covariant derivatives and finding solutions to the Bianchi identities.

This paper is organized as follows. In section 2, we construct a superspace geometry with the superconformal algebra as its structure group and impose a set of constraints on the curvatures analogous to the constraints imposed in the $\mathcal{N} = 1$ theory. In section 3, we describe in detail how the superconformal tensor calculus [5–10] corresponds to the component version of the superspace theory. In section 4, we demonstrate how the superconformal structure may be “degauged” to reproduce the $\mathcal{N} = 2$ conformal supergravity structure found by Howe in superspace [18]. In the conclusion, we briefly speculate about possible extensions of this formalism.

2 $\mathcal{N} = 2$ conformal superspace

In this section we present our construction of $\mathcal{N} = 2$ conformal superspace, the superspace structure which arises from the gauging of the $\mathcal{N} = 2$ superconformal algebra. In all of the salient details, the construction parallels our previous work on $\mathcal{N} = 1$ conformal superspace [21], but we will attempt to present the material in a self-contained way. We begin with defining the $\mathcal{N} = 2$ superconformal algebra to fix our notation. Next, we summarize the gauging procedure. Then we explain the curvature constraints which correspond to $\mathcal{N} = 2$ conformal supergravity. We finish with a summary of the $\mathcal{N} = 2$ supersymmetric action principles.

2.1 The global $\mathcal{N} = 2$ superconformal algebra

The global $\mathcal{N} = 2$ superconformal algebra can be constructed from the $\mathcal{N} = 2$ super-Poincaré algebra,

$$\begin{aligned}\{Q_\alpha{}^i, \bar{Q}_{\dot{\alpha}j}\} &= -2i\delta_j^i\sigma_{\alpha\dot{\alpha}}^a P_a \\ [M_{ab}, M_{cd}] &= \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} \\ [M_{ab}, P_c] &= P_a\eta_{bc} - P_b\eta_{ac} \\ [M_{ab}, Q_\gamma{}^i] &= (\sigma_{ab})_\gamma{}^\beta Q_\beta{}^i, \quad [M_{ab}, \bar{Q}^{\dot{\gamma}}{}_i] = (\bar{\sigma}_{ab})^{\dot{\gamma}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}{}_i\end{aligned}\tag{2.1}$$

⁴More accurately, the constraints were the usual integrability conditions for the existence of covariantly chiral superfields.

with all other commutators vanishing. The supersymmetry generator $Q_\alpha{}^i$ lies in the chiral spinor representation of the Lorentz group⁵ and in the isospinor representation of $SU(2)_R$. As usual, conjugate representations of the $SU(2)_R$ group are related by raising and lowering with the antisymmetric tensor ϵ_{ij} . Our conventions are

$$\epsilon^{\underline{12}} = \epsilon_{\underline{21}} = 1, \quad Q_{\alpha i} = \epsilon_{ij} Q_\alpha{}^j, \quad \bar{Q}_{\dot{\alpha}}{}^i = \epsilon^{ij} \bar{Q}_{\dot{\alpha} j}. \quad (2.2)$$

The bosonic part of this algebra can be extended to include the conformal algebra. This requires the introduction of the dilatation operator \mathbb{D} and the special conformal operator K_a . Consistency requires the further introduction of three new operators: the fermionic special conformal operator $S_{\alpha i}$ (and its conjugate $\bar{S}^{\dot{\alpha} i}$), as well as the $U(1)_R$ chiral rotation operator \mathbb{A} and the $SU(2)_R$ isospin operator $I_j^i = I_j{}^i$ which together span the R-symmetry group $U(2)_R$.

The special conformal and superconformal generators K and S have obvious Lorentz and $SU(2)_R$ transformation properties, while their dilatation and $U(1)_R$ properties are opposite those of P and Q :

$$\begin{aligned} [\mathbb{D}, P_a] &= P_a, & [\mathbb{D}, Q_\alpha{}^i] &= \frac{1}{2} Q_\alpha{}^i, & [\mathbb{D}, \bar{Q}_{\dot{\alpha}}{}^i] &= \frac{1}{2} \bar{Q}_{\dot{\alpha}}{}^i \\ [\mathbb{D}, K_a] &= -K_a, & [\mathbb{D}, S_{\alpha i}] &= -\frac{1}{2} S_{\alpha i}, & [\mathbb{D}, \bar{S}^{\dot{\alpha} i}] &= -\frac{1}{2} \bar{S}^{\dot{\alpha} i} \\ [\mathbb{A}, Q_\alpha{}^i] &= -i Q_\alpha{}^i, & [\mathbb{A}, \bar{Q}_{\dot{\alpha}}{}^i] &= +i \bar{Q}_{\dot{\alpha}}{}^i \\ [\mathbb{A}, S_{\alpha i}] &= +i S_{\alpha i}, & [\mathbb{A}, \bar{S}^{\dot{\alpha} i}] &= -i \bar{S}^{\dot{\alpha} i} \\ [I_j^i, Q_\alpha{}^k] &= \delta_j^k Q_\alpha{}^i - \frac{1}{2} \delta_j^i Q_\alpha{}^k, & [I_j^i, \bar{Q}_{\dot{\alpha}}{}^k] &= -\delta_k^i \bar{Q}_{\dot{\alpha}}{}^j + \frac{1}{2} \delta_j^i \bar{Q}_{\dot{\alpha}}{}^k \\ [I_j^i, S_{\alpha k}] &= -\delta_k^i S_{\alpha j} + \frac{1}{2} \delta_j^i S_{\alpha k}, & [I_j^i, \bar{S}^{\dot{\alpha} k}] &= \delta_j^k \bar{S}^{\dot{\alpha} i} - \frac{1}{2} \delta_j^i \bar{S}^{\dot{\alpha} k} \\ [M_{ab}, K_c] &= K_a \eta_{bc} - K_b \eta_{ac} \\ [M_{ab}, S_{\gamma i}] &= (\sigma_{ab})_\gamma{}^\beta S_{\beta i}. \end{aligned} \quad (2.3)$$

The special conformal generators have an algebra among each other that is similar to the supersymmetry algebra:

$$\{S_{\alpha i}, \bar{S}^{\dot{\alpha} j}\} = +2i \delta_i^j \sigma_{\alpha\dot{\alpha}}^a K_a. \quad (2.4)$$

Finally, the commutators of the special conformal generators with the translation and supersymmetry generators are

$$\begin{aligned} [K_a, P_b] &= 2\eta_{ab} \mathbb{D} - 2M_{ab} \\ \{S_{\alpha i}, Q_\beta{}^j\} &= 2\delta_i^j \epsilon_{\alpha\beta} \mathbb{D} - 2\delta_i^j M_{\alpha\beta} - i\delta_i^j \epsilon_{\alpha\beta} \mathbb{A} - 4\epsilon_{\alpha\beta} I_i{}^j \\ \{\bar{S}^{\dot{\alpha} i}, \bar{Q}_{\dot{\beta}}{}^j\} &= 2\delta_j^i \epsilon^{\dot{\alpha}\dot{\beta}} \mathbb{D} - 2\delta_j^i M^{\dot{\alpha}\dot{\beta}} + i\delta_j^i \epsilon^{\dot{\alpha}\dot{\beta}} \mathbb{A} + 4\epsilon^{\dot{\alpha}\dot{\beta}} I_j{}^i \\ [K_a, Q_\alpha{}^i] &= i\sigma_{a\alpha\dot{\beta}} \bar{S}^{\dot{\beta} i}, & [K_a, \bar{Q}_{\dot{\alpha}}{}^i] &= i\bar{\sigma}_a^{\dot{\alpha}\beta} S_{\beta i} \\ [S_{\alpha i}, P_a] &= i\sigma_{a\alpha\dot{\beta}} \bar{Q}_{\dot{\beta}}{}^i, & [\bar{S}^{\dot{\alpha} i}, P_a] &= i\bar{\sigma}_a^{\dot{\alpha}\beta} Q_{\beta}{}^i. \end{aligned} \quad (2.5)$$

⁵We use the same notation for representations of the Lorentz group as in [21] and [22]. For representations of the $SU(2)_R$ algebra, we follow the conventions of [19, 20].

All other commutators vanish.

We have made use of the convenient shorthand

$$\begin{aligned} M_{\alpha\beta} &\equiv -\epsilon_{\beta\gamma}(\sigma^{ba})_{\alpha}{}^{\gamma} M_{ab} = -(\sigma^{ba})_{\alpha\beta} M_{ab} \\ M^{\dot{\alpha}\dot{\beta}} &\equiv -\epsilon^{\dot{\beta}\dot{\gamma}}(\bar{\sigma}^{ba})^{\dot{\alpha}}{}_{\dot{\gamma}} M_{ab} = -(\bar{\sigma}^{ba})^{\dot{\alpha}\dot{\beta}} M_{ab}. \end{aligned} \quad (2.6)$$

These are projections of the Lorentz generator; $M_{\alpha\beta}$ rotates undotted spinors while $M^{\dot{\alpha}\dot{\beta}}$ rotates dotted spinors:

$$\begin{aligned} [M_{\alpha\beta}, Q_{\gamma}{}^i] &= -Q_{\alpha}{}^i \epsilon_{\beta\gamma} - Q_{\beta}{}^i \epsilon_{\alpha\gamma} \\ [M^{\dot{\alpha}\dot{\beta}}, \bar{Q}_{\dot{\gamma}}{}^i] &= -\bar{Q}_{\dot{\alpha}}{}^i \epsilon^{\dot{\beta}\dot{\gamma}} - \bar{Q}_{\dot{\beta}}{}^i \epsilon^{\dot{\alpha}\dot{\gamma}} \\ [M_{\alpha\beta}, P_{\gamma\dot{\gamma}}] &= -P_{\alpha\dot{\gamma}} \epsilon_{\beta\gamma} - P_{\beta\dot{\gamma}} \epsilon_{\alpha\gamma}, \quad P_{\gamma\dot{\gamma}} \equiv \sigma_{\gamma\dot{\gamma}}^c P_c. \end{aligned}$$

Note that for this definition of M ,

$$\frac{1}{2}\lambda^{ab}M_{ba} = \frac{1}{2}\lambda^{\alpha\beta}M_{\beta\alpha} + \frac{1}{2}\lambda_{\dot{\alpha}\dot{\beta}}M^{\dot{\beta}\dot{\alpha}} \quad (2.7)$$

using the decomposition rule⁶

$$\lambda_{\alpha\dot{\alpha}\beta\dot{\beta}} = 2\epsilon_{\dot{\alpha}\dot{\beta}}\lambda_{\alpha\beta} - 2\epsilon_{\alpha\beta}\lambda_{\dot{\alpha}\dot{\beta}}. \quad (2.8)$$

It is common for spinor indices to come paired with isospin indices, so we introduce boldface notation α to encompass both. For example, we can denote the supersymmetry generators by

$$Q_{\alpha} = Q_{\alpha}{}^i, \quad \bar{Q}^{\dot{\alpha}} = \bar{Q}^{\dot{\alpha}}{}_i.$$

Similarly, we write ξQ and $\bar{\xi}\bar{Q}$ as shorthand for

$$\xi^{\alpha}Q_{\alpha} = \xi^{\alpha}{}_i Q_{\alpha}{}^i, \quad \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} = \bar{\xi}_{\dot{\alpha}}{}^i \bar{Q}^{\dot{\alpha}}{}_i.$$

It is convenient to introduce the symbols

$$\begin{aligned} C_{\alpha\beta} &= \epsilon_{\alpha\beta}\epsilon^{ij}, & C^{\alpha\beta} &= \epsilon^{\alpha\beta}\epsilon_{ij} \\ C_{\dot{\alpha}\dot{\beta}} &= \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{ij}, & C^{\dot{\alpha}\dot{\beta}} &= \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{ij} \end{aligned} \quad (2.9)$$

for raising and lowering both sets of indices simultaneously; for example,

$$\xi_{\alpha} = C_{\alpha\beta}\xi^{\beta} = \xi_{\alpha}{}^i, \quad \bar{\xi}^{\dot{\alpha}} = C^{\dot{\alpha}\dot{\beta}}\bar{\xi}_{\dot{\beta}} = \bar{\xi}^{\dot{\alpha}}{}_i.$$

Similarly we may introduce $\delta_{\alpha}{}^{\beta}$ with the definition

$$\delta_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta}\delta_j^i, \quad \delta^{\dot{\alpha}}{}_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}}\delta_i^j.$$

It follows that

$$C_{\alpha\beta}C^{\beta\gamma} = \delta_{\alpha}{}^{\gamma}.$$

⁶This implies that $\lambda_{\alpha\beta}$ and $\lambda_{\dot{\alpha}\dot{\beta}}$ are anti-conjugate to each other using the rule for complex conjugation given in [3].

Finally, we mention one last convention. We symmetrize isospin and Lorentz indices with an appropriate factor of $n!$; for example,

$$V_{(i}W_{j)} = \frac{1}{2}V_iW_j + \frac{1}{2}V_jW_i, \quad V_{(\alpha}W_{\beta)} = \frac{1}{2}V_\alpha W_\beta + \frac{1}{2}V_\beta W_\alpha.$$

However, we *do not* include the $n!$ when *antisymmetrizing*. For example,

$$V_{[a}W_{b]} = V_aW_b - V_bW_a.$$

This is so that (among other reasons) we can write curvatures as $F_{nm} = \partial_{[n}A_{m]}$ without additional factors of 2. We follow the same convention when defining graded antisymmetrization: $F_{NM} = \partial_{[N}A_{M]} = \partial_N A_M - \partial_M A_N(-)^{nm}$.

2.2 $\mathcal{N} = 2$ superspace and gauging the $\mathcal{N} = 2$ superconformal algebra

The natural domain in which to deal with $\mathcal{N} = 2$ supersymmetry is the four dimensional $\mathcal{N} = 2$ superspace $\mathcal{M}^{4|8}$ parametrized by the local coordinates $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}}) = (x^m, \theta^\mu_i, \bar{\theta}_{\dot{\mu}}^i)$ where $m = 0, 1, 2, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$, and $i = \underline{1}, \underline{2}$. To encode supergravity, the geometry of $\mathcal{N} = 2$ superspace must be nontrivial; that is, we must have a nontrivial vielbein and non-vanishing connections. However, gauging the superconformal algebra is a less trivial task than gauging super Yang-Mills or even the Lorentz algebra. The reason for this is that the (graded) commutator of the special (super)conformal generator $K^A = (K^a, S^\alpha, \bar{S}_{\dot{\alpha}})$ with $P_A = (P_a, Q_\alpha, \bar{Q}^{\dot{\alpha}})$ gives generators other than P_A ; this means that other connections will, under the action of K , transform into the vielbein. Moreover, P_A cannot quite be the same as the covariant derivative since the (graded) commutator of P_A with itself corresponds to a flat geometry.⁷

Because we have a good number of generators, it will be useful to use a compact notation. We will denote the elements $\mathbb{D}, \mathbb{A}, I^i_j, M_{ab}, K_A$ by the generic notation $X_{\underline{a}}$. It should be emphasized that this set closes among itself under (graded) commutation; we denote this set \mathcal{H} . The remaining generators are the super-translation generators P_A . The algebra of all these elements may be written

$$[X_{\underline{a}}, X_{\underline{b}}] = -f_{\underline{ab}}^{\underline{c}} X_{\underline{c}} \tag{2.10a}$$

$$[X_{\underline{a}}, P_B] = -f_{\underline{aB}}^{\underline{c}} X_{\underline{c}} - f_{\underline{aB}}^C P_C \tag{2.10b}$$

$$[P_A, P_B] = -T_{AB}^C P_C \tag{2.10c}$$

where the f 's are structure constants and T_{AB}^C is the constant torsion tensor. We will *gauge* such an algebra by constructing an operator ∇_A , called the *covariant derivative*, which plays the role of P_A in the above relations, with the caveat that the last relation is relaxed to something more general:

$$[\nabla_A, \nabla_B] = -T_{AB}^C \nabla_C - R_{AB}^{\underline{c}} X_{\underline{c}} \tag{2.11}$$

⁷We will present a point of view of how to gauge such an algebra which seems to differ in interpretation from other approaches reviewed for example in [1]. (We elaborated more fully on our interpretation in [21].) However, it should be emphasized that the difference is *merely* interpretation; the final geometric structure arrived at is the same.

where $T_{AB}{}^C$ is a more general torsion tensor and $R_{AB}{}^c$ a curvature tensor.

We begin by associating a connection one-form $H_M{}^a$ with each generator X_a we wish to gauge. In addition, we introduce the vielbein one-form $E_M{}^A$ with the usual physical interpretation of equipping the manifold $\mathcal{M}^{4|8}$ with a tangent frame. Under the gauge transformations \mathcal{H} associated with X_a , we take

$$\delta_{\mathcal{H}} E_M{}^A = E_M{}^B g^c f_{cB}{}^A \quad (2.12a)$$

$$\delta_{\mathcal{H}} H_M{}^a = \partial_M g^a + E_M{}^B g^c f_{cB}{}^a + H_M{}^b g^c f_{cb}{}^a \quad (2.12b)$$

where $f_{cB}{}^A$ and $f_{cB}{}^a$ are the structure constants from (2.10).

A *covariant* superfield Ψ is defined by the property that it transforms under gauge transformations \mathcal{H} without any derivative on the parameter g^a ,

$$\delta_{\mathcal{H}} \Psi = g^a X_a \Psi. \quad (2.13)$$

The operator X_a acts on Ψ , transforming this local field into some other local field. For example, if Ψ is a conformally *primary* superfield, we have

$$K_A \Psi = 0, \quad (2.14)$$

whereas if Ψ is a *descendant* of some other fields Φ_A , we have

$$K_A \Psi = \Phi_A. \quad (2.15)$$

For the other generators, we normally have the usual matrix representations

$$\mathbb{D}\Psi = \Delta\Psi, \quad \mathbb{A}\Psi = iw\Psi, \quad I^i{}_j\Psi = \mathcal{J}^i{}_j\Psi, \quad M_{ab}\Psi = \mathcal{S}_{ab}\Psi, \quad (2.16)$$

where Δ and w are real numbers, corresponding to the conformal dimension and $U(1)_R$ weights of Ψ , while $\mathcal{J}^i{}_j$ and \mathcal{S}_{ab} are the isospin and Lorentz matrices associated with Ψ 's representation.

Because the parameter g^a is a local superfield, $\partial_M \Psi$ does not transform covariantly. We must introduce instead the *covariant derivative*

$$\nabla_A \Psi \equiv E_A{}^M \partial_M \Psi - E_A{}^M H_M{}^a X_a \Psi. \quad (2.17)$$

The superfield $\nabla_A \Psi$ is also covariant; one can show that it transforms as

$$\delta_{\mathcal{H}}(\nabla_A \Psi) = g^b \nabla_A X_b \Psi - g^b f_{bA}{}^C \nabla_C \Psi - g^b f_{bA}{}^c X_c \Psi \quad (2.18)$$

without any derivatives of g . If we denote $\delta_{\mathcal{H}}(\nabla_A \Psi) = g^b X_b \nabla_A \Psi$, we immediately find the operator relation

$$[X_b, \nabla_A] = -f_{bA}{}^C \nabla_C - f_{bA}{}^c X_c. \quad (2.19)$$

The *curvatures* of the manifold \mathcal{M} are found by taking the commutator of two covariant derivatives,

$$[\nabla_A, \nabla_B] \Psi = -T_{AB}{}^C \nabla_C \Psi - R_{AB}{}^c X_c \Psi \quad (2.20)$$

where T and R are the torsion and \mathcal{H} -curvature two-forms, respectively. They are most easily written as two-forms:

$$T^A \equiv dE^A - E^B \wedge H^c f_{cB}{}^A \quad (2.21a)$$

$$R^a \equiv dH^a - E^B \wedge H^c f_{cB}{}^a - \frac{1}{2} H^b \wedge H^c f_{cb}{}^a. \quad (2.21b)$$

Our structure can be understood then as consisting of a set of operators $X_{\underline{a}}$ and ∇_A with an algebra

$$[X_{\underline{a}}, X_{\underline{b}}] = -f_{\underline{ab}}{}^c X_{\underline{c}} \quad (2.22a)$$

$$[X_{\underline{a}}, \nabla_B] = -f_{\underline{a}B}{}^C \nabla_C - f_{\underline{a}B}{}^c X_{\underline{c}} \quad (2.22b)$$

$$[\nabla_A, \nabla_B] = -T_{AB}{}^C \nabla_C - R_{AB}{}^c X_{\underline{c}}. \quad (2.22c)$$

For the *global* superconformal algebra, we may identify ∇_A with P_A provided the \mathcal{H} -curvatures R and all torsions (except $T_{\alpha}{}^{\beta c}$) vanish. This is simply the geometry of flat superspace. A curved superspace corresponds then to a deformation of the superconformal algebra by introducing extra structure *functions* T and R into the algebra in the form of curvatures. This is the sense in which we “gauge” the superconformal algebra.

In order for this operator structure to be consistent, the Jacobi identities must be satisfied. We know they are satisfied for the global algebra, so we need to check them only for the local case. The Jacobi identities involving at most one ∇ are

$$0 = [X_{\underline{a}}, [X_{\underline{b}}, X_{\underline{c}}]] + \text{permutations}$$

$$0 = [X_{\underline{a}}, [X_{\underline{b}}, \nabla_C]] + \text{permutations}$$

and both are unchanged from the global case. The Jacobi identity involving two ∇ 's is

$$0 = [X_{\underline{a}}, [\nabla_B, \nabla_C]] + \text{permutations},$$

which implies that the curvatures transform under \mathcal{H} covariantly as

$$X_{\underline{a}} T_{BC}{}^D = -T_{BC}{}^F f_{F\underline{a}}{}^D - f_{\underline{a}[B}{}^F T_{FC]}{}^D - f_{\underline{a}[B}{}^f f_{fC]}{}^D \quad (2.23a)$$

$$X_{\underline{a}} R_{BC}{}^{\underline{d}} = -T_{BC}{}^F f_{F\underline{a}}{}^{\underline{d}} - R_{BC}{}^f f_{f\underline{a}}{}^{\underline{d}} - f_{\underline{a}[B}{}^F R_{FC]}{}^{\underline{d}} - f_{\underline{a}[B}{}^f f_{fC]}{}^{\underline{d}}. \quad (2.23b)$$

One can check using the definition of the curvatures, (2.21a) and (2.21b), along with the rules for the transformation of the connections, (2.12a) and (2.12b), that the curvatures do indeed transform in this way. Finally, the Jacobi identities involving three ∇ 's (i.e. the Bianchi identities) are

$$0 = [\nabla_A, [\nabla_B, \nabla_C]] + \text{permutations}. \quad (2.24)$$

We will check that these are satisfied in the next section once we postulate the form of the curvatures.

We collect our definitions of the curvatures and connections for the case of the superconformal algebra in Appendix B.

2.3 Conformal supergravity curvature constraints

As with all theories in superspace, the number of degrees of freedom in the connections far exceeds any reasonable number. We must therefore constrain the theory. The usual way this is done is by constraining curvatures (for super Yang-Mills theories) or torsions (for supergravity theories) and then (1) checking the Bianchi identity and (2) ensuring that the theory is not overconstrained.

The first of these is a difficult enough problem in superspace. Thankfully, we already have a clue as to how to proceed. In our previous work on $\mathcal{N} = 1$ conformal superspace [21], we found that the constraints necessary to encode conformal supergravity were a subset of “gauge” constraints – that is, constraints which superficially resemble the constraints of super Yang-Mills. The Bianchi identity was then very easy to check since its form was identical to the Bianchi identity of super Yang-Mills. It turns out that the same is true of $\mathcal{N} = 2$ conformal supergravity in superspace. We begin then by postulating the following form for the spinor derivative curvatures:

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = -2\epsilon^{ij}\epsilon_{\alpha\beta}\bar{\mathcal{W}} \quad (2.25a)$$

$$\{\bar{\nabla}^{\dot{\alpha}}_i, \bar{\nabla}^{\dot{\beta}}_j\} = +2\epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\mathcal{W} \quad (2.25b)$$

$$\{\nabla_\alpha^i, \bar{\nabla}^{\dot{\beta}}_j\} = -2i\delta_j^i\nabla_{\alpha\dot{\beta}} \quad (2.25c)$$

where \mathcal{W} and $\bar{\mathcal{W}}$ are valued in the superconformal algebra,

$$\mathcal{W} = \mathcal{W}(P)^A\nabla_A + \frac{1}{2}\mathcal{W}(M)^{ba}M_{ab} + \mathcal{W}(\mathbb{D})\mathbb{D} + \mathcal{W}(\mathbb{A})\mathbb{A} + \mathcal{W}(I)^j{}_i I^i{}_j + \mathcal{W}(K)^A K_A. \quad (2.26)$$

We will soon impose some of these \mathcal{W} to be zero, but for the moment we will consider this more general case. As in super Yang-Mills, the operator \mathcal{W} must obey two conditions in order for the Bianchi identities to be satisfied. First, \mathcal{W} must be a chiral operator⁸

$$[\nabla_\alpha^i, \bar{\mathcal{W}}] = [\bar{\nabla}^{\dot{\alpha}}_i, \mathcal{W}] = 0. \quad (2.27)$$

Second, it must obey the Bianchi identity

$$\{\nabla^{\phi i}, [\nabla_{\phi j}, \mathcal{W}]\} = \{\bar{\nabla}^{\dot{\phi} i}, [\bar{\nabla}_{\dot{\phi} j}, \bar{\mathcal{W}}]\}. \quad (2.28)$$

In order for the Jacobi identities to be satisfied, \mathcal{W} must obey one nontrivial constraint: it must be conformally primary

$$[S_{\beta i}, \mathcal{W}] = [S^{\dot{\beta} j}, \bar{\mathcal{W}}] = 0 \implies [K_c, \mathcal{W}] = 0. \quad (2.29)$$

It is straightforward to derive the form of all other curvatures in terms of the operator \mathcal{W} . For the dimension-3/2 curvatures, we have

$$[\nabla_\beta^i, \nabla_{\alpha\dot{\alpha}}] = -2\epsilon_{\beta\alpha}\bar{\mathcal{W}}_{\dot{\alpha}}^i \quad (2.30a)$$

$$[\bar{\nabla}_{\dot{\beta} i}, \nabla_{\alpha\dot{\alpha}}] = -2\epsilon_{\beta\alpha}\mathcal{W}_{\alpha i} \quad (2.30b)$$

⁸It is important to note that this condition does *not* mean that the components of \mathcal{W} are all chiral superfields.

where

$$\mathcal{W}_{\alpha i} \equiv -\frac{i}{2}\epsilon_{ij}[\nabla_{\alpha}^j, \mathcal{W}] \quad (2.31a)$$

$$\mathcal{W}_{\dot{\alpha}}^i \equiv -\frac{i}{2}\epsilon^{ij}[\bar{\nabla}_{\dot{\alpha}j}, \mathcal{W}]. \quad (2.31b)$$

The dimension-2 curvatures are

$$[\nabla_{\beta\dot{\beta}}, \nabla_{\alpha\dot{\alpha}}] = -\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}} = -2\epsilon_{\dot{\beta}\dot{\alpha}}\mathcal{F}_{\beta\alpha} + 2\epsilon_{\beta\alpha}\mathcal{F}_{\dot{\beta}\dot{\alpha}}, \quad (2.32)$$

where

$$\mathcal{F}_{\beta\alpha} = \frac{1}{8}\{\nabla_{(\beta}^k, [\nabla_{\alpha)k}, \mathcal{W}]\} \quad (2.33a)$$

$$\mathcal{F}_{\dot{\beta}\dot{\alpha}} = \frac{1}{8}\{\bar{\nabla}_{(\dot{\beta}}^k, [\bar{\nabla}_{\dot{\alpha})k}, \bar{\mathcal{W}}]\}. \quad (2.33b)$$

Now we must specify the precise form of \mathcal{W} and $\bar{\mathcal{W}}$. In the $\mathcal{N} = 1$ case, the spinor superfield \mathcal{W}_{α} was constrained so that only the Lorentz and special conformal components were nonvanishing; moreover, they were both given in terms of a single chiral superfield $W_{\alpha\beta\gamma}$, containing the field strengths of $\mathcal{N} = 1$ conformal supergravity. It turns out that an analogous structure may be imposed here. We take the scalar superfield \mathcal{W} to be defined in terms of the single chiral superfield $W_{\alpha\beta}$ and its derivatives,

$$\mathcal{W} = \frac{1}{2}W^{\alpha\beta}M_{\beta\alpha} + \mathcal{W}(S)^{\beta j}S_{\beta j} + \mathcal{W}(K)^b K_b \quad (2.34a)$$

$$\bar{\mathcal{W}} = \frac{1}{2}\bar{W}_{\dot{\alpha}\dot{\beta}}M^{\dot{\beta}\dot{\alpha}} + \bar{\mathcal{W}}(S)_{\dot{\beta}j}\bar{S}^{\beta j} + \bar{\mathcal{W}}(K)^b K_b \quad (2.34b)$$

where

$$\mathcal{W}(S)_{\alpha}^i = \frac{1}{4}\nabla^{\beta i}W_{\beta\alpha}, \quad \bar{\mathcal{W}}(S)_i^{\dot{\alpha}} = \frac{1}{4}\nabla_{\dot{\beta}i}\bar{W}^{\dot{\beta}\dot{\alpha}} \quad (2.35a)$$

$$\mathcal{W}(K)^{\dot{\alpha}\alpha} = -\frac{1}{2}\nabla^{\dot{\alpha}\beta}W_{\beta}^{\alpha}, \quad \bar{\mathcal{W}}(K)_{\alpha\dot{\alpha}} = -\frac{1}{2}\nabla_{\alpha\dot{\beta}}\bar{W}^{\dot{\beta}\dot{\alpha}}. \quad (2.35b)$$

It is straightforward to check that \mathcal{W} obeys all the necessary constraints provided the superfield $W_{\alpha\beta}$ is chiral, primary, and constrained to obey the Bianchi identity

$$\nabla^{\alpha\beta}W_{\beta\alpha} \equiv \nabla^{\alpha k}\nabla_k^{\beta}W_{\beta\alpha} = \nabla^{\dot{\alpha}k}\nabla_k^{\dot{\beta}}\bar{W}_{\dot{\beta}\dot{\alpha}} \equiv \bar{\nabla}^{\dot{\alpha}\dot{\beta}}\bar{W}_{\dot{\beta}\dot{\alpha}} \quad (2.36)$$

To show that our structure is indeed $\mathcal{N} = 2$ conformal supergravity can be done in two ways. The first is to show that when reduced to components, the curvatures and constraints on the curvatures are identical to those imposed in the $\mathcal{N} = 2$ superconformal tensor calculus, which is well known to describe conformal supergravity in components [5–7]. The second is to show that when the structure group is degauged to $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_{\text{R}}$, the theory reduces to Howe’s formulation of $\mathcal{N} = 2$ conformal supergravity [18]. We will show both, (re)constructing the superconformal tensor calculus in section 3 and Howe’s formulation in section 4.

We can now give the form of the remaining curvatures. The dimension-3/2 curvatures are given by the spinor superfield operators

$$\begin{aligned}\mathcal{W}_{\alpha j} = & \frac{i}{2}W_{\alpha}^{\gamma}\nabla_{\gamma j} - \frac{i}{4}\nabla^{\phi}{}_j W_{\phi\alpha}\mathbb{D} - \frac{1}{8}\nabla^{\phi}{}_j W_{\phi\alpha}\mathbb{A} + \frac{i}{2}\nabla^{\phi k}W_{\phi\alpha}I_{kj} - \frac{i}{4}\nabla^{\beta}{}_j W^{\gamma}{}_{\alpha}M_{\beta\gamma} \\ & - \frac{i}{8}\nabla_{\alpha j}\nabla^{\phi k}W_{\phi}^{\gamma}S_{\gamma k} + \frac{1}{4}\nabla_{\dot{\beta}}^{\phi}W_{\phi\alpha}\bar{S}^{\dot{\beta}}{}_j - \frac{i}{8}\nabla_{\alpha j}\nabla_{\dot{\beta}}^{\phi}W_{\phi\beta}K^{\dot{\beta}\beta}\end{aligned}\quad (2.37a)$$

$$\begin{aligned}\mathcal{W}^{\dot{\alpha}j} = & \frac{i}{2}\bar{W}^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\nabla}^{\dot{\gamma}j} - \frac{i}{4}\bar{\nabla}_{\dot{\phi}}^j\bar{W}^{\phi\dot{\alpha}}\mathbb{D} + \frac{1}{8}\bar{\nabla}_{\dot{\phi}}^j\bar{W}^{\phi\dot{\alpha}}\mathbb{A} - \frac{i}{2}\bar{\nabla}_{\dot{\phi}k}^j\bar{W}^{\phi\dot{\alpha}}I^{kj} - \frac{i}{4}\bar{\nabla}_{\dot{\beta}}^j\bar{W}_{\dot{\gamma}}{}^{\dot{\alpha}}M^{\dot{\beta}\dot{\gamma}} \\ & - \frac{i}{8}\bar{\nabla}^{\dot{\alpha}j}\nabla_{\dot{\phi}k}\bar{W}^{\phi}{}_{\dot{\gamma}}\bar{S}^{\dot{\gamma}k} + \frac{1}{4}\nabla^{\beta}{}_{\dot{\phi}}\bar{W}^{\phi\dot{\alpha}}S_{\beta}{}^j - \frac{i}{8}\bar{\nabla}^{\dot{\alpha}j}\nabla^{\beta}{}_{\dot{\phi}}\bar{W}^{\phi\dot{\beta}}K_{\beta\dot{\beta}}.\end{aligned}\quad (2.37b)$$

The dimension-2 curvatures are most easily given by specifying the components $\mathcal{F}_{\beta\alpha}$ and $\mathcal{F}_{\dot{\beta}\dot{\alpha}}$ of the curvature tensor $\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}$ given by

$$\mathcal{F}_{ba} = T_{ba}{}^C\nabla_C + \frac{1}{2}R_{ba}{}^{dc}M_{cd} + R(I)_{ba}{}^i{}_j I^j{}_i + R(\mathbb{D})_{ba}\mathbb{D} + R(\mathbb{A})_{ba}\mathbb{A} + R(K)_{ba}{}^CK_C. \quad (2.38)$$

The torsion is

$$T_{\beta\alpha}{}^C\nabla_C = +\frac{1}{4}\nabla^{\gamma}{}_j W_{\beta\alpha}\nabla_{\gamma}{}^j \quad (2.39a)$$

$$T_{\dot{\beta}\dot{\alpha}}{}^C\nabla_C = -\frac{1}{4}\bar{\nabla}_{\dot{\gamma}}{}^j \bar{W}_{\dot{\beta}\dot{\alpha}}\bar{\nabla}^{\dot{\gamma}}{}_j. \quad (2.39b)$$

The Lorentz curvature is

$$\frac{1}{2}R_{\beta\alpha}{}^{cd}M_{dc} = \frac{1}{16}\nabla^{\gamma\delta}W_{\beta\alpha}M_{\delta\gamma} - \frac{1}{16}\nabla_{\phi\gamma}W^{\phi\gamma}M_{\alpha\beta} - \frac{1}{4}W_{\alpha\beta}\bar{W}_{\dot{\gamma}\dot{\delta}}M^{\dot{\delta}\dot{\gamma}} \quad (2.40a)$$

$$\frac{1}{2}R_{\dot{\beta}\dot{\alpha}}{}^{cd}M_{dc} = \frac{1}{16}\bar{\nabla}_{\dot{\gamma}\dot{\delta}}\bar{W}_{\dot{\beta}\dot{\alpha}}M^{\dot{\delta}\dot{\gamma}} - \frac{1}{16}\nabla^{\gamma\phi}\bar{W}_{\dot{\phi}\dot{\gamma}}M_{\dot{\alpha}\dot{\beta}} - \frac{1}{4}\bar{W}_{\dot{\alpha}\dot{\beta}}W^{\gamma\delta}M_{\delta\gamma}. \quad (2.40b)$$

The isospin curvature is

$$R(I)_{\beta\alpha}{}^{ij} = -\frac{1}{8}\nabla^{ij}W_{\beta\alpha} \quad (2.41a)$$

$$R(I)_{\dot{\beta}\dot{\alpha}}{}^{ij} = -\frac{1}{8}\bar{\nabla}^{ij}\bar{W}_{\dot{\beta}\dot{\alpha}}. \quad (2.41b)$$

The dilatation curvatures are

$$R(\mathbb{D})_{\beta\alpha} = \frac{1}{16}\nabla_{\beta}{}^{\gamma}W_{\gamma\alpha} + \frac{1}{16}\nabla_{\alpha}{}^{\gamma}W_{\gamma\beta} \quad (2.42a)$$

$$R(\mathbb{D})_{\dot{\beta}\dot{\alpha}} = \frac{1}{16}\bar{\nabla}_{\dot{\beta}\dot{\gamma}}\bar{W}^{\dot{\gamma}}{}_{\dot{\alpha}} + \frac{1}{16}\bar{\nabla}_{\dot{\alpha}\dot{\gamma}}\bar{W}^{\dot{\gamma}}{}_{\dot{\beta}} \quad (2.42b)$$

and the $U(1)_R$ curvatures are

$$R(\mathbb{A})_{\beta\alpha} = -\frac{i}{32}\nabla_{\beta}{}^{\gamma}W_{\gamma\alpha} - \frac{i}{32}\nabla_{\alpha}{}^{\gamma}W_{\gamma\beta} \quad (2.43a)$$

$$R(\mathbb{A})_{\dot{\beta}\dot{\alpha}} = +\frac{i}{32}\bar{\nabla}_{\dot{\beta}\dot{\gamma}}\bar{W}^{\dot{\gamma}}{}_{\dot{\alpha}} + \frac{i}{32}\bar{\nabla}_{\dot{\alpha}\dot{\gamma}}\bar{W}^{\dot{\gamma}}{}_{\dot{\beta}}. \quad (2.43b)$$

They are related by $R(\mathbb{D})_{\beta\alpha} = 2iR(\mathbb{A})_{\beta\alpha}$ and $R(\mathbb{D})_{\dot{\beta}\dot{\alpha}} = -2iR(\mathbb{A})_{\dot{\beta}\dot{\alpha}}$. Finally, the special conformal curvatures are

$$\begin{aligned} R(K)_{\beta\alpha}{}^C K_C &= \frac{1}{32} \nabla_{\beta\alpha} \nabla^{\dot{\gamma}\delta} W_{\delta}{}^{\gamma} K_{\gamma\dot{\gamma}} + \frac{1}{96} \nabla^{ij} \nabla_{\beta i} W_{\alpha}{}^{\gamma} S_{\gamma j} + \frac{1}{96} \nabla^{ij} \nabla_{\alpha i} W_{\beta}{}^{\gamma} S_{\gamma j} \\ &\quad - \frac{i}{16} \nabla_{\beta j} \nabla^{\dot{\gamma}}{}^{\gamma} W_{\gamma\alpha} \bar{S}^{\dot{\gamma}j} - \frac{i}{16} \nabla_{\alpha j} \nabla^{\dot{\gamma}}{}^{\gamma} W_{\gamma\beta} \bar{S}^{\dot{\gamma}j} \\ &\quad - \frac{1}{8} W_{\beta\alpha} \bar{\nabla}_{\dot{\phi}j} \bar{W}^{\dot{\phi}}{}_{\dot{\gamma}} \bar{S}^{\dot{\gamma}j} - \frac{1}{8} W_{\beta\alpha} \nabla_{\gamma\dot{\phi}} \bar{W}^{\dot{\phi}}{}_{\dot{\gamma}} K^{\dot{\gamma}\gamma} \end{aligned} \quad (2.44a)$$

$$\begin{aligned} R(K)_{\dot{\beta}\dot{\alpha}}{}^C K_C &= \frac{1}{32} \bar{\nabla}_{\dot{\beta}\dot{\alpha}} \nabla_{\gamma\dot{\delta}} \bar{W}^{\dot{\delta}}{}_{\dot{\gamma}} K^{\dot{\gamma}\gamma} - \frac{1}{96} \bar{\nabla}_{ij} \bar{\nabla}_{\dot{\beta}}{}^j \bar{W}_{\dot{\alpha}\dot{\gamma}} \bar{S}^{\dot{\gamma}j} - \frac{1}{96} \bar{\nabla}_{ij} \bar{\nabla}_{\dot{\alpha}}{}^j \bar{W}_{\dot{\beta}\dot{\gamma}} \bar{S}^{\dot{\gamma}j} \\ &\quad + \frac{i}{16} \bar{\nabla}_{\dot{\beta}}{}^j \nabla^{\dot{\gamma}}{}_{\dot{\gamma}} \bar{W}^{\dot{\gamma}}{}_{\dot{\alpha}} S_{\gamma j} + \frac{i}{16} \bar{\nabla}_{\dot{\alpha}}{}^j \nabla^{\dot{\gamma}}{}_{\dot{\gamma}} \bar{W}^{\dot{\gamma}}{}_{\dot{\beta}} S_{\gamma j} \\ &\quad - \frac{1}{8} \bar{W}_{\dot{\beta}\dot{\alpha}} \nabla^{\phi j} W_{\phi}{}^{\gamma} S_{\gamma j} - \frac{1}{8} \bar{W}_{\dot{\beta}\dot{\alpha}} \nabla^{\dot{\gamma}\phi} W_{\phi}{}^{\gamma} K_{\gamma\dot{\gamma}}. \end{aligned} \quad (2.44b)$$

We summarize these relations (and give an alternative form for the special conformal curvatures) in Appendix B.

2.4 Superconformally invariant actions

Having derived the algebra of covariant derivatives, we turn next to the construction of superconformally invariant actions. Within the context of pure $\mathcal{N} = 2$ superspace (that is, without introducing an auxiliary \mathbb{CP}^1 or S^2 manifold as in projective [23–26] or harmonic [27, 28] superspace), there are two types of supersymmetric actions which may be constructed. The first involves an integral over the full superspace,

$$S = \int d^{12}z E V, \quad d^{12}z \equiv d^4x d^4\theta d^4\bar{\theta}. \quad (2.45)$$

Here $E = \text{sdet}(E_M{}^A)$ is the appropriate measure ensuring superdiffeomorphism invariance for a real scalar superfield Lagrangian V . The invariance of this action requires [21] that V transform trivially under each of the generators of the conformal algebra,

$$\mathbb{D}V = \mathbb{A}V = I_j^i V = M_{ab}V = K_a V = S_{\alpha} V = \bar{S}^{\dot{\alpha}} V = 0.$$

In other words, V must be a conformally primary Lorentz and $U(2)_R$ scalar with vanishing conformal dimension.

The second type of action available is the chiral action,

$$S_c = \int d^8\mathfrak{z} \mathcal{E} U + \text{h.c.}, \quad d^8\mathfrak{z} \equiv d^4x d^4\theta, \quad \bar{\nabla}^{\dot{\alpha}} U = 0. \quad (2.46)$$

The first term on the right involves an integration over the chiral manifold $\mathfrak{M}^{4|4}$ parametrized by the coordinates $\mathfrak{z}^{\mathfrak{m}} = (x^m, \theta^{\mu})$. We use the gothic index \mathfrak{m} to correspond to the set (m, μ) ; similarly, we use \mathfrak{a} for (a, α) . The space $\mathfrak{M}^{4|4}$ may be understood as a submanifold of $\mathcal{M}^{4|8}$, corresponding to the $\bar{\theta} = 0$ slice in the gauge where⁹

$$E_M{}^A = \begin{pmatrix} E_{\mathfrak{m}}{}^{\mathfrak{a}} & E_{\mathfrak{m}\dot{\alpha}} \\ E^{\dot{\mu}\mathfrak{a}} & E^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{\mathfrak{m}}{}^{\mathfrak{a}} & E_{\mathfrak{m}\dot{\alpha}} \\ 0 & \delta^{\dot{\mu}}{}_{\dot{\alpha}} \end{pmatrix}. \quad (2.47)$$

⁹We will soon show this gauge exists by direct construction.

The chiral measure \mathcal{E} is given by $\text{sdet}(\mathcal{E}_m^a)$.

In order for the action S_c to be invariant under the superconformal group, the chiral Lagrangian U must obey

$$\begin{aligned}\bar{\nabla}^{\dot{\alpha}}U &= 0, \quad \mathbb{D}U = 2U, \quad \mathbb{A}U = 4iU, \quad I^i{}_j U = M_{ab}U = 0 \\ K_a U &= S_{\alpha}U = \bar{S}^{\dot{\alpha}}U = 0.\end{aligned}$$

That is, U must be a conformally primary Lorentz and $\text{SU}(2)_R$ chiral scalar with conformal dimension 2 and $\text{U}(1)_R$ weight 4. In addition, the consistency conditions

$$\{\nabla^{\dot{\alpha}}, \nabla^{\dot{\beta}}\}U = 0$$

must be obeyed; for the curvatures we have imposed, we indeed have

$$0 = \{\nabla^{\dot{\alpha}}{}_i, \nabla^{\dot{\beta}}{}_j\}U = 2\epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{1}{2}W^{\gamma\delta}M_{\gamma\delta} + W(S)^{\gamma k}S_{\gamma k} + W(K)^c K_c\right)U \quad (2.48)$$

satisfied trivially since U is a Lorentz scalar and conformally primary. Similarly, we have the consistency conditions

$$0 = \{\bar{S}^{\dot{\alpha}i}, \bar{\nabla}^{\dot{\beta}}{}_j\}U = \delta_j^i(2\epsilon^{\dot{\alpha}\dot{\beta}}\mathbb{D} - 2M^{\dot{\alpha}\dot{\beta}} + i\epsilon^{\dot{\alpha}\dot{\beta}}\mathbb{A})U + 4\epsilon^{\dot{\alpha}\dot{\beta}}I^i{}_j U. \quad (2.49)$$

which are satisfied for scalar U with the given dilatation and $\text{U}(1)_R$ weights. Finally, one additional torsion constraint must be satisfied,¹⁰

$$0 = T^{\dot{\alpha}}{}_b{}^b - T^{\dot{\alpha}}{}_{\beta}{}^{\beta}, \quad (2.50)$$

which is obeyed trivially for our choice of torsion tensor.

2.4.1 Converting full superspace integrals to chiral integrals

As in $\mathcal{N} = 1$ superspace, one may convert an integral over full superspace to one over chiral superspace. The globally supersymmetric result is

$$\int d^{12}z V = \int d^8\mathfrak{z} \bar{D}^4 V, \quad \bar{D}^4 \equiv \frac{1}{48} \bar{D}^{ij} \bar{D}_{ij}. \quad (2.51)$$

We will prove that its locally supersymmetric generalization is simply

$$\int d^{12}z E V = \int d^8\mathfrak{z} \mathcal{E} \bar{\nabla}^4 V, \quad \bar{\nabla}^4 \equiv \frac{1}{48} \bar{\nabla}^{ij} \bar{\nabla}_{ij}. \quad (2.52)$$

The first step to evaluating the integral is to construct a certain normal coordinate system where the covariant derivative $\bar{\nabla}^{\dot{\alpha}}$ is especially simple. Given the generic form

$$\bar{\nabla}^{\dot{\alpha}} = E^{\dot{\alpha}M} \left(\partial_M - H_M{}^b X_b \right) \quad (2.53)$$

for the covariant derivative, along with the constraints

$$\{\bar{\nabla}^{\dot{\alpha}}, \bar{\nabla}^{\dot{\beta}}\} = 2C^{\dot{\alpha}\dot{\beta}} \mathcal{W}^c X_c, \quad [\bar{\nabla}^{\dot{\alpha}}, \mathcal{W}^c X_c] = 0, \quad (2.54)$$

¹⁰For a proof of this and previous results, see section A.2.5 of [21].

one can adopt the normal coordinate gauge

$$E^{\dot{\mu}A} = \delta^{\dot{\mu}A}, \quad H^{\dot{\mu}\underline{a}} = -C^{\dot{\mu}\dot{\nu}}\bar{\theta}_{\dot{\nu}}\mathcal{W}^{\underline{a}}. \quad (2.55)$$

This gauge choice requires the fixing of $\bar{\theta}$ -dependent terms in all of the gauge degrees of freedom (including diffeomorphisms). In particular, this gauge choice for the vielbein is identical to the gauge discussed in (2.47). For the superconformal group, the covariant derivative $\bar{\nabla}^{\dot{\alpha}}$ then takes the especially simple form

$$\bar{\nabla}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} - \frac{1}{2}\Omega^{\dot{\alpha}\beta\gamma}M_{\gamma\beta} - F^{\dot{\alpha}b}K_b - F^{\dot{\alpha}\beta j}S_{\beta j}. \quad (2.56)$$

For a primary scalar field Ψ , it follows that $\bar{\nabla}^{\dot{\alpha}}\Psi = \bar{\partial}^{\dot{\alpha}}\Psi$, $\bar{\nabla}^{\dot{\beta}}\bar{\nabla}^{\dot{\alpha}}\Psi = \bar{\partial}^{\dot{\beta}}\bar{\partial}^{\dot{\alpha}}\Psi$, and so on.

This gauge choice implies

$$E = \text{sdet}(E_M{}^A) = \text{sdet}(\mathcal{E}_m{}^a) = \mathcal{E}, \quad (2.57)$$

and one may show that E is independent of $\bar{\theta}$,

$$\partial^{\dot{\mu}}E = \nabla^{\dot{\mu}}E = E(\nabla^{\dot{\mu}}E_N{}^A)E_A{}^N(-)^n = E(T^{\dot{\mu}}{}_A{}^A(-)^a + \nabla_A E^{\dot{\mu}A}) = 0. \quad (2.58)$$

It immediately follows that in this gauge

$$\int d^{12}z E V = \int d^8\mathfrak{z} \mathcal{E} \frac{1}{48} \bar{\partial}^{ij} \bar{\partial}_{ij} V \quad (2.59)$$

and since in this gauge we have $\bar{\partial}^{ij} \bar{\partial}_{ij} V = \bar{\nabla}^{ij} \bar{\nabla}_{ij} V$, we arrive at the expression

$$\int d^{12}z E V = \int d^8\mathfrak{z} \mathcal{E} \frac{1}{48} \bar{\nabla}^{ij} \bar{\nabla}_{ij} V \quad (2.60)$$

in this particular gauge. The result (2.52), valid in any gauge, follows from gauge invariance of both sides of the above relation; that is, because it holds in the specific gauge we have constructed here, it must hold in all gauges. This is a suspiciously simple-looking result; we will give greater credibility to it in section 4.5 where we show that it implies the more complicated chiral projection operator familiar from Howe's and Grimm's formulations of $\mathcal{N} = 2$ superspace as constructed by Müller [29].

In $\mathcal{N} = 1$ Poincaré superspace, it is possible to perform this process in reverse, converting an integral over the chiral superspace to one over the whole superspace using the relation

$$\int d^4\mathfrak{z} \mathcal{E} U = \frac{1}{2} \int d^8z \frac{E}{R} U. \quad (2.61)$$

This result follows from the $\mathcal{N} = 1$ conformal superspace relation

$$\int d^4\mathfrak{z} \mathcal{E} U = -4 \int d^8z E \frac{X}{\bar{\nabla}^2 X} U \quad (2.62)$$

where X is a real superfield of conformal dimension two. In this expression, $\bar{\nabla}^2 X$ is chiral and primary and so the right hand side is gauge invariant. If one degauges the

conformally covariant derivative to the normal Poincaré covariant derivative, one must make the replacement

$$\bar{\nabla}^2 X \longrightarrow (\bar{D}^2 - 8R)X.$$

If one simultaneously gauges X to unity using the dilatational gauge symmetry, the result (2.61) follows.

It is natural to ask whether an analogous relation may be constructed for $\mathcal{N} = 2$ superspace. Although we haven't yet discussed how the degauging procedure works, it is easy to see that the answer must be no. The analogue of (2.62) is

$$\int d^8\mathfrak{z} \mathcal{E} U = \int d^{12}z E \frac{Z}{\bar{\nabla}^4 Z} U. \quad (2.63)$$

However, in order for $\bar{\nabla}^4 Z$ to be both chiral and primary, Z must be of dimension *zero*. This prevents us from adopting the gauge $Z = 1$ and so no analogue of (2.61) is possible¹¹, at least within the superspace framework discussed here.¹²

2.4.2 Component actions

It remains to derive component actions from $\mathcal{N} = 2$ actions in superspace. Since any $\mathcal{N} = 2$ action over the full superspace may be written as an integral over chiral superspace, the problem is, as in $\mathcal{N} = 1$, the derivation of the component form of a generic chiral action. Within $\mathcal{N} = 2$ conformal supergravity, this component action was constructed originally via the superconformal tensor calculus [6]. We give here a direct superspace construction of the same result.¹³

One begins with the action

$$S = \int d^8\mathfrak{z} \mathcal{E} U = \int d^4x e \mathcal{L}_c \quad (2.64)$$

where U is a conformally primary chiral superfield of weight $(2, 4)$; our goal is to evaluate the component chiral Lagrangian \mathcal{L}_c . This can be done most simply by first going to the specific θ -dependent gauge, where

$$E_\mu{}^A = \delta_\mu{}^A, \quad H_\mu{}^a = C_{\mu\nu} \theta^\nu \bar{\mathcal{W}}^a. \quad (2.65)$$

Because of the constraints we have placed on $\bar{\mathcal{W}}$, the spinor covariant derivative in this gauge involves only three connection terms,

$$\nabla_\alpha = \partial_\alpha - \frac{1}{2} \Omega_{\alpha\dot{\beta}\gamma} \bar{M}^{\dot{\gamma}\dot{\beta}} - F_\alpha{}^b K_b - F_{\alpha\dot{\beta}j} \bar{S}^{\dot{\beta}j}. \quad (2.66)$$

¹¹This is usually understood by noting that the chiral projection operator in conventional $\mathcal{N} = 2$ superspace descriptions (4.41) annihilates any constant scalar field.

¹²It is possible to construct something like (2.61) if superspace is augmented to include an internal \mathbb{CP}^1 manifold. The construction is given in [30].

¹³The construction for conventional $\mathcal{N} = 2$ superspace was given for the first time by Müller [31].

In this gauge, $\mathcal{E} = e \equiv \det(e_m^a)$ has nontrivial θ -dependence, so the component Lagrangian takes the form

$$\begin{aligned} e\mathcal{L}_c &= \frac{1}{48} \partial^{ij} \partial_{ij} (eU) \\ &= \frac{1}{48} e \partial^{ij} \partial_{ij} U + \frac{1}{12} \partial^{\alpha i} e \partial_{\alpha}^j \partial_{ij} U + \frac{1}{16} \partial^{ij} e \partial_{ij} U - \frac{1}{16} \partial^{\alpha\beta} e \partial_{\alpha\beta} U \\ &\quad + \frac{1}{12} \partial^{\alpha i} \partial_{ij} e \partial_{\alpha}^j U + \frac{1}{48} \partial^{ij} \partial_{ij} e U \end{aligned} \quad (2.67)$$

where we have defined

$$\partial^{ij} \equiv \partial^{\beta i} \partial_{\beta}^j, \quad \partial_{\alpha\beta} \equiv \partial_{\alpha}^j \partial_{\beta j}. \quad (2.68)$$

Because of the restricted form of (2.66), one may replace $\partial_{\alpha}^i \longrightarrow \nabla_{\alpha}^i$ in each of the above terms. For the derivatives acting on U , this gives terms involving covariant derivatives of U , which cannot be further simplified without considering a specific form for U . However, the derivatives acting on e must be simplified further. One can proceed either by explicitly constructing the θ -expansion of e , or by making use of the torsion relations recursively, which amounts to the same. We take the second approach since it is easier in application.

There are two relations which we will need. The first is

$$\nabla_{\alpha}^i e = e \nabla_{\alpha}^i e_m^b e_b^m = e T_{\alpha}^i{}^m{}_b e_b^m = ie (\sigma^b \bar{\psi}_b^i)_{\alpha}.$$

The second is

$$\begin{aligned} \nabla_{\alpha}^i \bar{\psi}_{c\dot{\beta}}^j &= (\nabla_{\alpha}^i e_c^m) \bar{\psi}_{m\dot{\beta}}^j + e_c^m \nabla_{\alpha}^i \bar{\psi}_{m\dot{\beta}}^j \\ &= -e_c^n T_{\alpha}^i{}^n{}_b \bar{\psi}_{b\dot{\beta}}^j + 2e_c^m T_{\alpha}^i{}^m{}_b \bar{\psi}_{b\dot{\beta}}^j \\ &= -i(\sigma^b \bar{\psi}_c^i)_{\alpha} \bar{\psi}_{b\dot{\beta}}^j + i\epsilon^{ij} \sigma_{c\alpha\dot{\gamma}} \bar{W}^{\dot{\gamma}}_{\dot{\beta}}. \end{aligned}$$

Applying these relations repeatedly allows us to evaluate all of the required spinor derivatives of e :

$$\begin{aligned} \nabla^{ij} e &= 4e (\bar{\psi}_m^i \bar{\sigma}^{mn} \bar{\psi}_n^j) \\ \nabla_{\alpha\beta} e &= -2e (\bar{\psi}_m \bar{\psi}_n) (\sigma^{mn})_{\alpha\beta} \\ \nabla_{\beta}^i \nabla^{\beta\alpha} e &= -\nabla_{\beta}^{\alpha} \nabla^j \bar{\psi}^j e = -3e \epsilon^{abcd} (\bar{\psi}_a \bar{\psi}_b) (\sigma_c \bar{\psi}_d^i)^{\alpha} + 6ie \bar{\sigma}_b^{\dot{\gamma}\alpha} \bar{W}_{\dot{\gamma}\dot{\beta}} \bar{\psi}_b^{\dot{\beta}i} \\ \nabla^{ij} \nabla_{ij} e &= -12ie \epsilon^{abcd} (\bar{\psi}_a \bar{\psi}_b) (\bar{\psi}_c \bar{\psi}_d) + 48 (\bar{\psi}_a \bar{\psi}_b) (\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\beta}\dot{\alpha}} + 48 \bar{W}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\beta}\dot{\alpha}} \end{aligned}$$

where we have defined $(\bar{\psi}_a \bar{\psi}_b) \equiv \bar{\psi}_a^j \bar{\psi}_{bj}$. Applying these results to (2.67) gives

$$\begin{aligned} \mathcal{L}_c &= \nabla^4 U - \frac{i}{12} (\bar{\psi}_m^i \bar{\sigma}^m)^{\alpha} \nabla_{\alpha}^j \nabla_{ij} U \\ &\quad + \frac{1}{4} (\bar{\psi}_m^i \bar{\sigma}^{mn} \bar{\psi}_n^j) \nabla_{ij} U + \frac{1}{8} (\bar{\psi}_m \bar{\psi}_n) (\sigma^{mn})^{\alpha\beta} \nabla_{\alpha\beta} U \\ &\quad + \frac{1}{4} \epsilon^{mnpq} (\bar{\psi}_m \bar{\psi}_n) (\bar{\psi}_p^i \bar{\sigma}_q)^{\alpha} \nabla_{\alpha i} U + \frac{i}{2} \bar{\psi}_{b\dot{\gamma}}^j \bar{W}^{\dot{\gamma}}_{\dot{\beta}} \bar{\sigma}_b^{\dot{\beta}\alpha} \nabla_{\alpha j} U \\ &\quad - \frac{i}{4} \epsilon^{mnpq} (\bar{\psi}_m \bar{\psi}_n) (\bar{\psi}_p \bar{\psi}_q) U + (\bar{\psi}_m \bar{\psi}_n) (\bar{\sigma}^{mn})^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} U + \bar{W}^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}\dot{\beta}} U. \end{aligned} \quad (2.69)$$

The action constructed from this Lagrangian is automatically $\mathcal{N} = 2$ superconformally invariant.

3 Component analysis

In section 2, we have derived the superspace geometry corresponding to conformal supergravity in superspace in a manner which *manifestly* respects the superconformal symmetry. In order for this to be the correct superspace theory, it must reduce in components to the $\mathcal{N} = 2$ superconformal tensor calculus, which reproduces the structure of $\mathcal{N} = 2$ conformal supergravity. The field content of $\mathcal{N} = 2$ conformal supergravity – the so-called “Weyl multiplet” – has been known for a long time [5, 7]. It consists first of a set of one-forms: the vierbein e_m^a , the gravitino $\psi_m^{\alpha i}$, the $U(1)_R$ and $SU(2)_R$ gauge fields A_m and $\phi_m^i{}_j$, and the dilatation gauge field b_m . The last of these, b_m , transforms algebraically under special conformal transformations and may be gauged away. In addition there are gauge connections associated with the rest of the superconformal group: the spin connection ω_m^{ab} and the special conformal and superconformal connections f_m^a and $\phi_m^{\alpha i}$, which are algebraically constrained to be functions of other fields.

The story so far is similar to the $\mathcal{N} = 1$ picture; however, counting degrees of freedom, one finds that the number of fermionic degrees of freedom do not match the number of bosonic ones. One is led to introduce additional “matter” fields: a fermion $\chi_{\alpha i}$, an anti-symmetric tensor W_{ab} , and a real scalar D . Taking these into account, one finds 24 bosonic and 24 fermionic degrees of freedom. This set of fields was first identified in [5, 7, 32].¹⁴

Our goal is to show that our construction in superspace reduces in components to that of [5–10]. The superfields available are the one-forms associated with each of the generators of the superconformal algebra as well as the superfield $W_{\alpha\beta}$ and its various spinorial derivatives. The component one-forms of the superconformal tensor calculus obviously come from projections of the corresponding superspace one-forms:

$$\begin{aligned} e_m^a &\equiv E_m^a|, & \psi_m^{\alpha i} &\equiv 2E_m^{\alpha i}|, & \bar{\psi}_{m\dot{\alpha}}^i &\equiv 2E_{m\dot{\alpha}}^i| \\ A_m &\equiv A_m|, & b_m &\equiv B_m|, & \omega_m^{ab} &\equiv \Omega_m^{ab}|, & \phi_m^i{}_j &\equiv \Phi_m^i{}_j| \\ f_m^a &\equiv F_m^a|, & \phi_m^{\alpha i} &\equiv 2F_m^{\alpha i}|, & \bar{\phi}_{m\dot{\alpha}i} &\equiv 2F_{m\dot{\alpha}i}|. \end{aligned} \quad (3.1)$$

In contrast, the “matter” fields of the $\mathcal{N} = 2$ Weyl multiplet must be identified with certain components of the superfield $W_{\alpha\beta}$ and its conjugate.¹⁵ The lowest components of each have the correct transformation properties to correspond to the self-dual and anti-self-dual parts of W_{ba} :

$$W_{ba} = W_{ba}^+ + W_{ba}^-, \quad W_{ba}^+ \equiv (\sigma_{ba})_{\beta}{}^{\alpha} W_{\alpha}{}^{\beta}|, \quad W_{ba}^- \equiv (\bar{\sigma}_{ba})^{\dot{\beta}}{}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}{}_{\dot{\beta}}| \quad (3.2)$$

$$\frac{i}{2} \epsilon_{dc}{}^{ba} W_{ba}^{\pm} = \pm W_{dc}^{\pm}. \quad (3.3)$$

¹⁴The linearized $SO(2)$ supergravity was constructed earlier in [33] and [34]. This model was identified in [5] as the Weyl multiplet coupled to a vector and a nonlinear compensator. It is worth noting that the Weyl multiplet is usually coupled to a vector compensator to generate the so-called “minimal field representation” with 32 bosonic and 32 fermionic components [14]. A second compensator is needed to give sensible equations of motion.

¹⁵This identification seems to have first been made at the linearized level by Bergshoeff, de Roo, and de Wit [32] who dealt with the general case of $\mathcal{N} \leq 4$, extending the earlier groundbreaking work of Ferrara and Zumino [35] for $\mathcal{N} = 1$.

The fermion $\chi_{\alpha i}$ and its conjugate correspond to

$$\chi_{\alpha i} = -\frac{1}{3}\nabla^\beta{}_i W_{\beta\alpha}|, \quad \bar{\chi}^{\dot{\alpha}i} = \frac{1}{3}\bar{\nabla}_{\dot{\beta}}{}^i \bar{W}^{\dot{\beta}\dot{\alpha}}|. \quad (3.4)$$

Finally, the scalar D is given by

$$D = \frac{1}{12}\nabla^{\alpha\beta} W_{\beta\alpha}| = \frac{1}{12}\bar{\nabla}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\beta}\dot{\alpha}}|, \quad (3.5)$$

which is real by virtue of the Bianchi identity (2.36) obeyed by $W_{\alpha\beta}$.

The other components of $W_{\alpha\beta}$ must then correspond to functions of the gauge and “matter” fields. This is certainly possible; recall that *all* components of the $\mathcal{N} = 1$ conformal superspace curvature $W_{\alpha\beta\gamma}$ are functions of the $\mathcal{N} = 1$ conformal supergravity connections. Furthermore, the constraints on the curvatures in superspace must lead to the same curvature constraints found in the STC formalism when reduced to components.

To understand how the reduction to components occurs, we will briefly review the case of globally supersymmetric Yang-Mills. At the component level, this theory consists of the gauge connection \mathbf{A}_m , the gaugino $\lambda_{\alpha i}$, a scalar field ϕ , and an auxiliary isotriplet \mathbf{D}^{ij} , each of which is in the adjoint of the gauge group; for example, $\mathbf{A}_m = A_m{}^r \mathbf{T}_r$ where \mathbf{T}_r is the generator in the adjoint. Within superspace, the theory consists of a gauge connection \mathbf{A}_M whose field strength is constrained in terms of a reduced chiral superfield \mathbf{W} . Only the vector part of the superfield \mathbf{A}_M contributes at the component level, whereas the rest of the Yang-Mills multiplet, the “matter” fields, are found within the chiral superfield \mathbf{W} via

$$\phi \propto \mathbf{W}|, \quad \lambda_{\alpha i} \propto \mathcal{D}_{\alpha i} \mathbf{W}|, \quad \mathbf{D}^{ij} \propto \mathcal{D}^{ij} \mathbf{W}| = \bar{\mathcal{D}}^{ij} \bar{\mathbf{W}}|, \quad (3.6)$$

with the precise coefficients depending on one’s conventions. There remains the component $\mathcal{D}_{\alpha\beta} \mathbf{W}$ and its conjugate, which are not independent fields. Rather, they are fixed by the vector field strength constraint

$$\mathbf{F}_{ba} = -\frac{1}{8}(\sigma_{ba})^{\beta\alpha} \mathcal{D}_{\alpha\beta} \mathbf{W}| - \frac{1}{8}(\bar{\sigma}_{ba})^{\dot{\beta}\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} \bar{\mathbf{W}}| \quad (3.7)$$

where

$$\mathbf{F}_{ba} \equiv e_b{}^n e_a{}^m \mathbf{F}_{nm} = e_b{}^n e_a{}^m (\partial_{[n} \mathbf{A}_{m]} - [\mathbf{A}_n, \mathbf{A}_m]) \quad (3.8)$$

For supergravity, the constraint analogous to (3.7) for the field strength is

$$\mathcal{F}_{ba} = -(\sigma_{ba})^{\beta\alpha} \mathcal{F}_{\alpha\beta} - (\bar{\sigma}_{ba})^{\dot{\beta}\dot{\alpha}} \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}} \quad (3.9)$$

where in conformal superspace

$$\mathcal{F}_{\alpha\beta} \equiv \frac{1}{8}\{\nabla_{(\alpha}{}^k, [\nabla_{\beta)k}, \mathcal{W}]\}, \quad \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}} \equiv \frac{1}{8}\{\bar{\nabla}_{(\dot{\alpha}}{}^k, [\bar{\nabla}_{\dot{\beta)k}}, \bar{\mathcal{W}}]\}. \quad (3.10)$$

The analogue of the definition (3.8) is more complicated in supergravity. One begins with

$$\begin{aligned} \mathcal{F}_{nm} = & E_n{}^b E_m{}^a \mathcal{F}_{ba} + E_{[n}{}^\beta E_{m]}{}^a \mathcal{F}_{\beta a} + E_{[n\dot{\beta}} E_{m]}{}^a \mathcal{F}^{\dot{\beta}}{}_a \\ & - E_n{}^\beta E_m{}^\alpha \mathcal{F}_{\beta\alpha} - E_{n\dot{\beta}} E_{m\dot{\alpha}} \mathcal{F}^{\dot{\beta}\dot{\alpha}} - E_{[n}{}^\beta E_{m]\dot{\alpha}} \mathcal{F}_\beta{}^{\dot{\alpha}} \end{aligned} \quad (3.11)$$

where as before we use a condensed summation convention for the bold indices α and $\dot{\alpha}$:

$$\chi^\alpha \lambda_\alpha \equiv \chi^\alpha{}_i \lambda_\alpha{}^i, \quad \bar{\chi}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \equiv \bar{\chi}_{\dot{\alpha}}{}^i \bar{\lambda}^{\dot{\alpha}}{}_i.$$

Projecting to lowest components and solving for $\mathcal{F}_{ba}|$ yields

$$\begin{aligned} \mathcal{F}_{ba}| &= e_b{}^n e_a{}^m \mathcal{F}_{nm} - \frac{1}{2}(\psi_{[b} \sigma_{a]} \bar{\lambda}) + \frac{1}{2}(\bar{\psi}_{[b} \bar{\sigma}_{a]} \lambda) \\ &\quad - \frac{1}{2}(\psi_b \psi_a) \bar{\mathcal{W}}| - \frac{1}{2}(\bar{\psi}_b \bar{\psi}_a) \mathcal{W}| - \frac{i}{2}(\psi_{[b} \sigma^c \bar{\psi}_{a]}) \nabla_c \end{aligned} \quad (3.12)$$

where we have applied the gauge constraints of conformal superspace (2.25) and made use of the suppressed index conventions

$$(\psi_n \sigma_m \bar{\lambda}) = \psi_n{}^\alpha{}_j \sigma_{m\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}j}, \quad (\bar{\psi}_n \bar{\sigma}_m \lambda) = \bar{\psi}_n{}^{\dot{\alpha}}{}_j \bar{\sigma}_m{}^{\alpha\dot{\alpha}} \lambda_{\alpha j} \quad (3.13)$$

$$(\psi_n \psi_m) = \psi_n{}^{\alpha j} \psi_{m\alpha j}, \quad (\bar{\psi}_n \bar{\psi}_m) = \bar{\psi}_n{}^{\dot{\alpha} j} \bar{\psi}_{m\dot{\alpha} j} \quad (3.14)$$

along with the definitions

$$\lambda_{\alpha j} \equiv \mathcal{W}_{\alpha j}| = -\frac{i}{2}[\nabla_{\alpha j}, \mathcal{W}|], \quad \bar{\lambda}^{\dot{\alpha} j} \equiv \bar{\mathcal{W}}^{\dot{\alpha} j}| = -\frac{i}{2}[\bar{\nabla}^{\dot{\alpha} j}, \bar{\mathcal{W}}|]. \quad (3.15)$$

We may take the relations (3.12) as the *definitions* of $\mathcal{F}_{ba}|$ in analogy to (3.8). These correspond to the so-called covariantized curvatures of the superconformal tensor calculus. The constraints imposed on $\mathcal{F}_{ba}|$ from (3.10) will then correspond to the constraints found by de Wit et al.

3.1 Comparison to superconformal tensor calculus

In order to make comparisons between our results and those of the superconformal tensor calculus, some conversions of notation are necessary. For definiteness, we use the more recent conventions of [11–13]. The primary difference is that de Wit et al. make use of four component spinors. Their Lorentz tensor conventions can be exchanged for ours by the replacements

$$\gamma^a \rightarrow \begin{pmatrix} 0 & i\sigma^a \\ i\bar{\sigma}^a & 0 \end{pmatrix}, \quad \gamma^{ab} \rightarrow \begin{pmatrix} -2\sigma^{ab} & 0 \\ 0 & -2\bar{\sigma}^{ab} \end{pmatrix}, \quad \epsilon_{abcd} \rightarrow -i\epsilon_{abcd}. \quad (3.16)$$

Second, their convention for SU(2) indices is opposite ours. Lowered indices must be raised and vice-versa. Third, there are some differing normalization conventions for several of the gauge fields. Exchanging their notation for ours necessitates making the replacements

$$A_m \rightarrow -2A_m \quad (3.17a)$$

$$\mathcal{V}_m{}^i{}_j \rightarrow +2\phi_{mi}{}^j \quad (3.17b)$$

$$\phi_{m\alpha j} \rightarrow -2\phi_{m\alpha}{}^j \quad (3.17c)$$

$$f_m{}^a \rightarrow -2f_m{}^a. \quad (3.17d)$$

Additionally, the antisymmetric two-form of the Weyl multiplet in STC carries antisymmetric $SU(2)$ indices (which is natural for $\mathcal{N} \geq 2$), whereas we use a notation specialized to $\mathcal{N} = 2$. They are related by

$$T_{ab}{}^{ij} \rightarrow -2\epsilon_{ij} W_{ab}^+. \quad (3.18)$$

Finally, there are two major differences in the definition of curvatures. The overall normalizations for $R(Q)_{cb}{}^i$ and $R(S)_{cbi}$ in STC differ from the corresponding curvatures in our approach by a factor of two. That is,

$$\begin{aligned} R(Q)_{cb}{}^i &= e_c{}^n e_b{}^m \mathcal{D}_{[n} \psi_m]{}^i + \dots \\ R(S)_{cbi} &= e_c{}^n e_b{}^m \mathcal{D}_{[n} \phi_m]_i + \dots \end{aligned}$$

whereas we have

$$\begin{aligned} T_{cb}{}^{\alpha}{}_i &= \frac{1}{2} e_c{}^n e_b{}^m \mathcal{D}_{[n} \psi_m]{}^{\alpha}{}_i + \dots \\ R(S)_{cb}{}^{\alpha i} &= \frac{1}{2} e_c{}^n e_b{}^m \mathcal{D}_{[n} \phi_m]{}^{\alpha i} + \dots \end{aligned}$$

Taking these definitions as well as the gauge field normalizations into account, curvatures in the STC approach may be related to our component curvatures via the replacements

$$R(P)_{cb}{}^a \rightarrow T_{cb}{}^a \quad (3.19a)$$

$$R(Q)_{cb}{}^{\alpha i} \rightarrow 2T_{cb}{}^{\alpha}{}_i \quad (3.19b)$$

$$R(M)_{dc}{}^{ba} \rightarrow R(M)_{dc}{}^{ba} \quad (3.19c)$$

$$R(A)_{ba} \rightarrow -2R(\mathbb{A})_{ba} \quad (3.19d)$$

$$R(D)_{ba} \rightarrow R(\mathbb{D})_{ba} \quad (3.19e)$$

$$R(\mathcal{V})_{ba}{}^i{}_j \rightarrow 2R(I)_{ba}{}^i{}_j \quad (3.19f)$$

$$R(K)_{cb}{}^a \rightarrow -2R(K)_{cb}{}^a \quad (3.19g)$$

$$R(S)_{cb}{}^{\alpha}{}_i \rightarrow -4R(S)_{cb}{}^{\alpha i}. \quad (3.19h)$$

The constraint structure of the superconformal tensor calculus can be briefly summarized in the following way. One introduces connections and curvatures for the entire superconformal algebra. The curvatures are then “covariantized” using the matter fields of the $\mathcal{N} = 2$ Weyl multiplet and the covariantized curvatures are constrained by three relations, which in our notation read

$$T_{cb}{}^a = 0 \quad (3.20)$$

$$T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}^{\beta}{}_i = -\frac{3}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\chi^{\gamma i} \quad (3.21)$$

$$R^c{}_{bca} = R(\mathbb{D})_{ba} + \frac{3a}{2}\eta_{ba}D - \eta^{cd}W_{ac}^+W_{bd}^- \quad (3.22)$$

These constraints are *algebraic* when describing the components; they fix the spin connection $\omega_m{}^{ab}$, the special superconformal connection $\phi_m{}^{\alpha i}$, and the special conformal connection $f_m{}^a$, respectively. The parameter a in the last expression is purely a matter of

convention; it has no effect on the physics, merely adding a term to the connection f_m^a and simultaneously removing it elsewhere in any given action.¹⁶ It turns out that our approach will correspond to the choice $a = -2$.

It is easy to see why the choice $a = -2$ has been made. In the commutator $[\delta_Q(\xi), \delta_{\bar{Q}}(\bar{\xi})]$ constructed from the tensor calculus, one finds [6]

$$[\delta_Q(\xi), \delta_{\bar{Q}}(\bar{\xi})] = -2i \delta_P(\xi \sigma^a \bar{\xi}) - \frac{i}{4}(a+2) \delta_K(\xi \sigma^a \bar{\xi} D). \quad (3.23)$$

This can be understood as arising from the superspace anticommutator

$$\{\nabla_\alpha^i, \bar{\nabla}_{\dot{\alpha}j}\} = -2i \delta_j^i \nabla_{\alpha\dot{\alpha}} - \frac{i}{48} \delta_j^i (a+2) \nabla^{\gamma\beta} W_{\gamma\beta} K_{\alpha\dot{\alpha}} \quad (3.24)$$

Our choice $a = -2$ clearly simplifies the algebra, but any other value is equally allowed; it simply imposes a conventional constraint on the curvature

$$R(K)_\alpha^{\dot{\beta}c} = \frac{1}{96}(a+2) T_\alpha^{\dot{\beta}c} \nabla^{\gamma\beta} W_{\gamma\beta}. \quad (3.25)$$

The component curvatures have a quite intricate structure, and identically obey a number of component Bianchi identities. We will evaluate each of their component forms and analyze the corresponding constraints imposed.

3.2 Torsion analysis

We begin by analyzing the components of the torsion tensor. Recall that T_{nm}^a is given in lowest components by

$$T_{nm}^a = \partial_{[n} e_{m]}^a + \omega_{[nm]}^a + b_{[n} e_{m]}^a. \quad (3.26)$$

Using the other components of the superspace torsion tensor, this can be covariantized to

$$T_{cb}^a = e_c^{\ n} e_b^{\ m} \partial_{[n} e_{m]}^a + \omega_{[cb]}^a + b_{[c} \delta_{b]}^a - \frac{i}{2}(\psi_{[c} \sigma^a \bar{\psi}_{b]}). \quad (3.27)$$

This quantity is constrained to vanish,

$$T_{cb}^a = 0, \quad (3.28)$$

which allows the spin connection ω_m^{ab} to be solved in terms of the vierbein and the gravitino:

$$\begin{aligned} \omega_{mnp} = e_n^{\ a} e_p^{\ b} \omega_{mab} = & \frac{1}{2}(e_m^{\ a} \partial_n e_{pa} - e_p^{\ a} \partial_m e_{na} - e_n^{\ a} \partial_p e_{ma}) \\ & - \frac{1}{2}(e_m^{\ a} \partial_p e_{na} - e_n^{\ a} \partial_m e_{pa} - e_p^{\ a} \partial_n e_{ma}) \\ & + \frac{i}{4}(\psi_p \sigma_m \bar{\psi}_n - \psi_m \sigma_n \bar{\psi}_p - \psi_n \sigma_p \bar{\psi}_m) \\ & - \frac{i}{4}(\psi_n \sigma_m \bar{\psi}_p - \psi_m \sigma_p \bar{\psi}_n - \psi_p \sigma_n \bar{\psi}_m) \\ & + b_n g_{pm} - b_p g_{nm}. \end{aligned} \quad (3.29)$$

This agrees with the conventional definition in supergravity, except for the explicit appearance of the dilatation connection b_m .

¹⁶The choice $a = 1$ is frequently made, especially in recent work [11–13].

3.3 Gravitino torsion analysis

The gravitino torsion tensor is given by

$$T_{nm}{}^{\alpha}{}_i = \frac{1}{2}\mathcal{D}_{[n}\psi_{m]}{}^{\alpha}{}_i + \frac{i}{2}(\bar{\phi}_{[ni}\bar{\sigma}_{m]})^{\alpha}, \quad (3.30)$$

where

$$\mathcal{D}_n\psi_m{}^{\alpha}{}_i \equiv \partial_n\psi_m{}^{\alpha}{}_i + \psi_m{}^{\beta}{}_i\omega_{n\beta}{}^{\alpha} - iA_n\psi_m{}^{\alpha}{}_i + \phi_{ni}{}^j\psi_m{}^{\alpha}{}_j + \frac{1}{2}b_n\psi_m{}^{\alpha}{}_i. \quad (3.31)$$

We have introduced the derivative \mathcal{D} , which contains the Lorentz, $U(2)_R$, and dilatation connections. The gravitino torsion may be covariantized to

$$T_{cb}{}^{\alpha}{}_i = \frac{1}{2}\psi_{cb}{}^{\alpha}{}_i + \frac{i}{2}(\bar{\phi}_{[ci}\bar{\sigma}_{b]})^{\alpha} + \frac{i}{8}\epsilon_{ik}(\bar{\psi}_{[c}{}^k\bar{\sigma}_{b]}\sigma^{fd})^{\dot{\gamma}\alpha}W_{fd}^{+} \quad (3.32)$$

$$\psi_{cb}{}^{\alpha}{}_i = e_c{}^ne_c{}^m\psi_{nm}{}^{\alpha}{}_i \equiv e_c{}^ne_c{}^m\mathcal{D}_{[n}\psi_{m]}{}^{\alpha}{}_i. \quad (3.33)$$

From superspace we have the relation

$$T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}^{\alpha}{}_i = \frac{1}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla^{\alpha}{}_iW_{\gamma\beta}| \quad (3.34)$$

which implies the constraint

$$T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}^{\beta}{}_i = -\frac{3}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\chi_{\gamma i}. \quad (3.35)$$

This allows the determination of $\bar{\phi}_{m\dot{\alpha}j}$:

$$\begin{aligned} \bar{\phi}_{\beta\dot{\beta}\dot{\alpha}j} = & -\frac{i}{6}\mathcal{D}_{\dot{\beta}}{}^{\phi}\psi_{\phi\dot{\alpha}\beta j} - \frac{i}{3}\mathcal{D}_{\dot{\alpha}}{}^{\phi}\psi_{\phi\dot{\beta}\beta j} - \frac{i}{6}\epsilon_{\dot{\beta}\dot{\alpha}}\mathcal{D}_{\beta}{}^{\dot{\phi}}\psi_{\phi\dot{\phi}}{}^{\phi}{}_j \\ & - \frac{1}{6}W_{\beta}{}^{\phi}\bar{\psi}_{\phi\dot{\beta}\dot{\alpha}j} - \frac{1}{3}W_{\beta}{}^{\phi}\bar{\psi}_{\phi\dot{\alpha}\dot{\beta}j} + \frac{i}{6}\epsilon_{\dot{\beta}\dot{\alpha}}\nabla^{\phi}{}_jW_{\phi\beta}| \end{aligned} \quad (3.36)$$

which may equivalently be written

$$\begin{aligned} \bar{\phi}_{m\dot{\alpha}j} = & \frac{i}{2}\left(\bar{\sigma}^{pn}\bar{\sigma}_m - \frac{1}{3}\bar{\sigma}_m\sigma^{pn}\right)^{\dot{\alpha}\beta}\left(\mathcal{D}_p\psi_{n\beta j} - \frac{i}{4}W_{ab}^{+}(\sigma^{ab}\sigma_p\bar{\psi}_{nj})_{\beta}\right) - \frac{i}{4}(\bar{\sigma}_m\chi_j)^{\dot{\alpha}} \\ = & \frac{i}{2}\left(\bar{\sigma}^{pn}\bar{\sigma}_m - \frac{1}{3}\bar{\sigma}_m\sigma^{pn}\right)^{\dot{\alpha}\beta}\mathcal{D}_p\psi_{n\beta j} - \frac{1}{3}W_{mn}^{+}\bar{\psi}^n{}_{\dot{\alpha}j} - \frac{1}{3}W_{mb}^{+}(\bar{\sigma}^{bn}\bar{\psi}_{nj})_{\dot{\alpha}} - \frac{i}{4}(\bar{\sigma}_m\chi_j)^{\dot{\alpha}}. \end{aligned} \quad (3.37)$$

Applying this result to (3.34) determines the spin-3/2 part of the spinor derivative of $W_{\alpha\beta}$

$$\sum_{(\alpha\beta\gamma)}\nabla_{\gamma}{}^jW_{\beta\alpha}| = \sum_{(\alpha\beta\gamma)}\left(\mathcal{D}_{\beta}{}^{\dot{\phi}}\psi_{\dot{\phi}\alpha}{}^j - i\bar{\psi}_{\beta\dot{\phi}}{}^{\dot{\phi}j}W_{\alpha\gamma}\right). \quad (3.38)$$

The sum is over all permutations of the indices. This relation is analogous to the definition of $W_{\alpha\beta\gamma}$ in $\mathcal{N} = 1$ supergravity.

For the conjugate formulae, one finds

$$\begin{aligned}\phi_{\beta\dot{\beta}}\alpha^j &= +\frac{i}{6}\mathcal{D}_{\beta}^{\dot{\phi}}\bar{\psi}_{\alpha\dot{\phi}}\dot{\beta}^j + \frac{i}{3}\mathcal{D}_{\alpha}^{\dot{\phi}}\bar{\psi}_{\beta\dot{\phi}}\dot{\beta}^j + \frac{i}{6}\epsilon_{\beta\alpha}\mathcal{D}_{\dot{\beta}}^{\phi}\psi_{\phi\dot{\phi}}^{\dot{\beta}j} \\ &\quad - \frac{1}{6}\bar{W}_{\dot{\beta}}^{\dot{\phi}}\psi_{\beta\dot{\phi}}\alpha^j - \frac{1}{3}\bar{W}_{\dot{\beta}}^{\dot{\phi}}\psi_{\alpha\dot{\phi}}\dot{\beta}^j - \frac{i}{6}\epsilon_{\beta\alpha}\bar{\nabla}^{\dot{\phi}j}\bar{W}_{\dot{\phi}\dot{\beta}}|\end{aligned}\quad (3.39)$$

equivalently written as

$$\begin{aligned}\phi_{m\alpha}^j &= \frac{i}{2}\left(\sigma^{pn}\sigma_m - \frac{1}{3}\sigma_m\bar{\sigma}^{pn}\right)_{\alpha\dot{\beta}}\left(\mathcal{D}_p\bar{\psi}_n^{\dot{\beta}j} + \frac{i}{4}W_{ab}^-(\bar{\sigma}^{ab}\bar{\sigma}_p\psi_n^j)^{\dot{\beta}}\right) - \frac{i}{4}(\sigma_m\bar{\chi}^j)_{\alpha} \\ &= \frac{i}{2}\left(\sigma^{pn}\sigma_m - \frac{1}{3}\sigma_m\bar{\sigma}^{pn}\right)_{\alpha\dot{\beta}}\left(\mathcal{D}_p\bar{\psi}_n^{\dot{\beta}j} + \frac{1}{3}W_{mn}^-\psi_n^{\alpha j} + \frac{1}{3}W_{mb}^-(\sigma^{bn}\psi_n^j)_{\alpha} - \frac{i}{4}(\sigma_m\bar{\chi}^j)_{\alpha}\right),\end{aligned}\quad (3.40)$$

along with the relation

$$\sum_{(\dot{\alpha}\dot{\beta}\dot{\gamma})}\bar{\nabla}_{\dot{\gamma}}^j\bar{W}_{\dot{\beta}\dot{\alpha}} = \sum_{(\dot{\alpha}\dot{\beta}\dot{\gamma})}\left(\mathcal{D}_{\dot{\beta}}^{\phi}\bar{\psi}_{\phi\dot{\alpha}}\dot{\gamma}^j + i\psi_{\phi\dot{\beta}}^{\phi j}\bar{W}_{\dot{\alpha}\dot{\gamma}}\right).\quad (3.41)$$

3.4 Dilatation and $U(1)_R$ curvatures

The dilatation and axial curvatures are given by

$$R(\mathbb{D})_{nm} = \partial_{[n}b_{m]} + 2f_{[nm]} + \frac{1}{2}\psi_{[n}\phi_{m]} + \frac{1}{2}\bar{\psi}_{[n}\bar{\phi}_{m]}\quad (3.42)$$

$$R(\mathbb{A})_{nm} = \partial_{[n}A_{m]} - \frac{i}{4}\psi_{[n}\phi_{m]} + \frac{i}{4}\bar{\psi}_{[n}\bar{\phi}_{m]}.\quad (3.43)$$

Their supercovariant forms are

$$R(\mathbb{D})_{ba} = e_b{}^ne_a{}^m\partial_{[n}b_{m]} + 2f_{[ba]} + \frac{1}{2}\psi_{[b}\phi_{a]} + \frac{1}{2}\bar{\psi}_{[b}\bar{\phi}_{a]} + \frac{3i}{8}(\psi_{[b}\sigma_a]\bar{\chi}) + \frac{3i}{8}(\bar{\psi}_{[b}\bar{\sigma}_a]\chi)\quad (3.44)$$

$$R(\mathbb{A})_{ba} = e_b{}^ne_a{}^m\partial_{[n}A_{m]} - \frac{i}{4}\psi_{[b}\phi_{a]} + \frac{i}{4}\bar{\psi}_{[b}\bar{\phi}_{a]} - \frac{3}{16}(\psi_{[b}\sigma_a]\bar{\chi}) + \frac{3}{16}(\bar{\psi}_{[b}\bar{\sigma}_a]\chi)\quad (3.45)$$

where we have used the suppressed index convention

$$(\psi_b\sigma_a\bar{\chi}) = (\psi_{b j}\sigma_a\bar{\chi}^j), \quad (\bar{\psi}_b\bar{\sigma}_a\chi) = (\bar{\psi}_b{}^j\sigma_a\chi_j)\quad (3.46)$$

with both sets of isospin indices in their natural positions.

However, these are constrained by

$$R(\mathbb{D})_{\beta\alpha} = 2iR(\mathbb{A})_{\beta\alpha} = \frac{1}{16}\nabla_{\beta}^{\gamma}W_{\gamma\alpha} + \frac{1}{16}\nabla_{\alpha}^{\gamma}W_{\gamma\beta}\quad (3.47)$$

$$R(\mathbb{D})_{\dot{\beta}\dot{\alpha}} = -2iR(\mathbb{A})_{\dot{\beta}\dot{\alpha}} = \frac{1}{16}\bar{\nabla}_{\dot{\beta}}^{\dot{\gamma}}\bar{W}_{\dot{\gamma}\dot{\alpha}} + \frac{1}{16}\bar{\nabla}_{\dot{\alpha}}^{\dot{\gamma}}\bar{W}_{\dot{\gamma}\dot{\beta}}.\quad (3.48)$$

These relations accomplish two things: they fix $\nabla_{(\beta}^{\gamma}W_{\alpha)\gamma}|$ and also require

$$R(\mathbb{D})_{dc} = -\epsilon_{dc}{}^{ba}R(\mathbb{A})_{ba}.\quad (3.49)$$

This latter condition in principle imposes a relation on $f_{[dc]}$; but, as we will shortly see, $f_m{}^a$ is *entirely* determined by a constraint on the Lorentz curvature just as in the superconformal tensor calculus. The above result is really a consistency condition and is identically satisfied as in the superconformal tensor calculus.

3.5 Lorentz curvature

The Lorentz curvature is given by

$$R_{nm}{}^{ba} = \partial_{[n}\omega_{m]}{}^{ba} - \omega_{[n}{}^{bc}\omega_{m]c}{}^a + 2e_{[n}{}^{[b}f_{m]}{}^{a]} + (\psi_{[n}\sigma^{ba}\phi_{m]}) + (\bar{\psi}_{[n}\bar{\sigma}^{ba}\bar{\phi}_{m]}) \quad (3.50)$$

where we have used the shorthand

$$(\psi_n\sigma^{ba}\phi_m) = (\psi_n{}^j\sigma^{ba}\phi_m{}_j), \quad (\bar{\psi}_n\bar{\sigma}^{ba}\bar{\phi}_m) = (\bar{\psi}_n{}^j\bar{\sigma}^{ba}\bar{\phi}_m{}_j) \quad (3.51)$$

with the isospin indices in their natural positions.

It will be useful to introduce the symbol $\hat{R}_{nm}{}^{ba}$ to correspond to the Poincaré version of the Lorentz curvature; that is, it equals the above expression without $f_m{}^a$ and $\phi_m{}^{\alpha i}$. The covariantized Lorentz curvature is

$$\begin{aligned} R_{dc}{}^{ba} = & \hat{R}_{dc}{}^{ba} + 2\delta_{[d}{}^{[b}f_{c]}{}^{a]} + (\psi_{[d}\sigma^{ba}\phi_{c]}) + (\bar{\psi}_{[d}\bar{\sigma}^{ba}\bar{\phi}_{c]}) \\ & - \frac{1}{2}(\psi_{[d}\sigma_{c]}\bar{\lambda}^{ba}) + \frac{1}{2}(\bar{\psi}_{[d}\bar{\sigma}_{c]}\lambda^{ba}) - \frac{1}{2}\psi_d\psi_c W^{ba-} - \frac{1}{2}\bar{\psi}_d\bar{\psi}_c W^{ba+} \end{aligned} \quad (3.52)$$

where we have used

$$(\psi_d\psi_c) = (\psi_d{}^j\psi_{cj}), \quad (\bar{\psi}_d\bar{\psi}_c) = (\bar{\psi}_d{}^j\bar{\psi}_{cj}). \quad (3.53)$$

The fermionic combination $\lambda_{\alpha j}{}^{ba}$ is equal to a certain combination of spinor derivatives of $W_{\alpha\beta}$; it may equivalently be written

$$\lambda_{\alpha j}{}^{ba} = i(\sigma^{dc}\sigma^{ba}T_{dcj}). \quad (3.54)$$

Making use of the constraint (3.35), this can be rewritten as

$$\lambda_{\alpha j}{}^{ba} = -2iT^{ba}{}_{\alpha j} + \frac{3i}{2}(\sigma^{ba}\chi_j)_{\alpha}, \quad (3.55)$$

which allows the final form of the covariantized Lorentz curvature to be written

$$\begin{aligned} R_{dc}{}^{ba} = & \hat{R}_{dc}{}^{ba} + 2\delta_{[d}{}^{[b}f_{c]}{}^{a]} + (\psi_{[d}\sigma^{ba}\phi_{c]}) + (\bar{\psi}_{[d}\bar{\sigma}^{ba}\bar{\phi}_{c]}) - i(\psi_{[d}\sigma_{c]}\bar{T}^{ba}) - i(\bar{\psi}_{[d}\bar{\sigma}_{c]}T^{ba}) \\ & + \frac{3i}{4}(\psi_{[d}\sigma_{c]}\bar{\sigma}^{ba}\bar{\chi}) + \frac{3i}{4}(\bar{\psi}_{[d}\bar{\sigma}_{c]}\sigma^{ba}\chi) - \frac{1}{2}\psi_d\psi_c W^{ba-} - \frac{1}{2}\bar{\psi}_d\bar{\psi}_c W^{ba+}. \end{aligned} \quad (3.56)$$

Because R_{dcba} , when written in spinor form, involves the totally symmetric combination $\nabla_{(\delta\gamma}W_{\beta\alpha)}$, the above relation determines the lowest component of this superfield.

However, the lower spin parts of R_{dcba} are constrained. Contracting indices to construct the Ricci tensor, we find

$$\mathcal{R}_{(\gamma\dot{\gamma})(\alpha\dot{\alpha})} = \frac{1}{4}\epsilon_{\dot{\gamma}\dot{\alpha}}\nabla_{(\gamma}^{\phi}W_{\alpha)\phi} - \frac{1}{4}\epsilon_{\gamma\alpha}\bar{\nabla}_{\dot{\phi}(\dot{\gamma}}\bar{W}_{\dot{\alpha})}^{\dot{\phi}} + \frac{1}{2}\epsilon_{\gamma\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}(\nabla^{\delta\beta}W_{\beta\delta}) + 2W_{\gamma\alpha}W_{\dot{\gamma}\dot{\alpha}}. \quad (3.57)$$

All of the terms on the right hand side have already been determined. In vector form, this relation can be written

$$\mathcal{R}_{ba} = R(\mathbb{D})_{ba} - 3\eta_{ba}D - \eta^{cd}W_{ac}^+W_{bd}^- \quad (3.58)$$

which is precisely the constraint we wanted to show for the choice $a = -2$. This allows the determination of f_m^a as in the superconformal tensor calculus. Usually one needs only f_a^a , which is given by

$$\begin{aligned} f_a^a = & -D - \frac{1}{12}\hat{\mathcal{R}} - \frac{1}{24}\epsilon^{abcd}(\bar{\psi}_a\bar{\sigma}_b\mathcal{D}_c\psi_d) + \frac{1}{24}\epsilon^{abcd}(\psi_a\sigma_b\mathcal{D}_c\bar{\psi}_d) \\ & - \frac{i}{8}(\psi_b\sigma^b\bar{\chi}) - \frac{i}{8}(\bar{\psi}_b\bar{\sigma}^b\chi) + \frac{1}{12}W^{ab+}(\bar{\psi}_a\bar{\psi}_b) + \frac{1}{12}W^{ab-}(\psi_a\psi_b) \end{aligned} \quad (3.59)$$

where we have defined

$$(\psi_a\sigma_b\mathcal{D}_c\bar{\psi}_d) = (\psi_{aj}\sigma_b\mathcal{D}_c\bar{\psi}_d^j), \quad (\bar{\psi}_a\bar{\sigma}_b\mathcal{D}_c\psi_d) = (\bar{\psi}_a^j\bar{\sigma}_b\mathcal{D}_c\psi_{dj}) \quad (3.60)$$

with the isospin indices in their natural positions.

3.6 Isospin curvature

The isospin curvature is given by

$$R(I)_{nmij} = \partial_{[n}\phi_{m]}_{ij} + \phi_{[n(i}{}^k\phi_{m]j)k} - \psi_{[n(i}\phi_{m]j)} + \bar{\psi}_{[n(i}\bar{\phi}_{m]j)}. \quad (3.61)$$

Its covariantized form is

$$\begin{aligned} R(I)_{baij} = & e_b{}^ne_a{}^m\partial_{[n}\phi_{m]}_{ij} + \phi_{[b(i}{}^k\phi_{a]j)k} - \psi_{[b(i}\phi_{a]j)} + \bar{\psi}_{[b(i}\bar{\phi}_{a]j)} \\ & - \frac{3i}{4}\psi_{[b(i}\sigma_{a]}\bar{\chi}_{j)} + \frac{3i}{4}\bar{\psi}_{[b(i}\bar{\sigma}_{a]}\chi_{j)}. \end{aligned} \quad (3.62)$$

This is constrained by the superspace structure to obey

$$R(I)_{\beta\dot{\beta}\alpha\dot{\alpha}ij} = -\frac{1}{4}\epsilon_{\beta\dot{\beta}}\nabla_{ij}W_{\beta\alpha}| + \frac{1}{4}\epsilon_{\beta\alpha}\bar{\nabla}_{ij}\bar{W}_{\dot{\beta}\dot{\alpha}}|. \quad (3.63)$$

This superspace constraint serves only to define $\nabla_{ij}W_{\beta\alpha}|$ and its conjugate; there is no constraint on the component curvature.

3.7 Special superconformal curvature

The special superconformal curvature is given by

$$R(S)_{nm}{}^{\alpha j} = \frac{1}{2}\mathcal{D}_{[n}\phi_{m]}{}^{\alpha j} - \frac{i}{2}(\bar{\psi}_{[n}{}^j\bar{\sigma}_c)^\alpha f_{m]}{}^c. \quad (3.64)$$

Its covariantized form is rather complicated. We begin with the defining relation,

$$R(S)_{ba}{}^{\alpha j} = e_b{}^ne_a{}^m R(S)_{nm}{}^{\alpha j} - \frac{1}{2}(\psi_{[b}\sigma_{a]}\bar{\lambda}^{\alpha j}) + \frac{1}{2}(\bar{\psi}_{[b}\bar{\sigma}_{a]}\lambda^{\alpha j}) + \frac{3}{8}(\bar{\psi}_{[b}\bar{\psi}_{a]})\chi^{\alpha j} \quad (3.65)$$

where we have made use of the relations

$$\mathcal{W}(S)^{\alpha j} = \frac{1}{4}\nabla^{\beta j}W_{\beta}{}^{\alpha}| = -\frac{3}{4}\chi^{\alpha j}, \quad \bar{\mathcal{W}}(S)^{\alpha j} = 0. \quad (3.66)$$

The expressions for λ and $\bar{\lambda}$ are rather complicated. The simpler is $\bar{\lambda}$, which can be written

$$\bar{\lambda}^{\dot{\beta}k\alpha j} = \frac{1}{4}\nabla_{\dot{\phi}}{}^{\alpha}\bar{W}^{\dot{\phi}\dot{\beta}}|\epsilon^{kj} = \frac{1}{8}\nabla_c\bar{W}_{ba}{}^{-}(\bar{\sigma}_{ba}\bar{\sigma}_c)^{\dot{\beta}\alpha}\epsilon^{kj}. \quad (3.67)$$

The expression for λ , on the other hand, is more intricate:

$$\lambda_{\beta k}{}^{\alpha j} = -\frac{i}{8}\nabla_{\beta k}\nabla^{\phi j}W_{\phi}{}^{\alpha}|. \quad (3.68)$$

Decomposing its Lorentz and isospin structures into irreducible representations, we find

$$\lambda_{\beta k}{}^{\alpha j} = \frac{i}{16}\nabla_j{}^k W_{\beta}{}^{\alpha}| + \frac{i}{16}\epsilon^{\alpha\gamma}\nabla_{(\beta}\nabla^{\phi}_{\gamma)}W_{\phi}{}^{\alpha}| - \frac{i}{32}\nabla^{\delta\phi}W_{\delta\phi}\delta_k{}^j\delta_{\beta}{}^{\alpha}|. \quad (3.69)$$

Each of these structures has already been specified, leading to

$$\lambda_{\beta k}{}^{\alpha j} = \frac{i}{4}R(I)_{ba}{}_j{}^k(\sigma^{ba})_{\beta}{}^{\alpha} + \frac{1}{2}\delta_k^j R(\mathbb{A})_{ba}(\sigma^{ba})_{\beta}{}^{\alpha} - \frac{3i}{8}D\delta_k{}^j\delta_{\beta}{}^{\alpha}. \quad (3.70)$$

This yields the covariantized form

$$\begin{aligned} R(S)_{ba}{}^{\alpha j} &= \frac{1}{2}e_b{}^ne_a{}^m\mathcal{D}_{[n}\phi_{m]}{}^{\alpha j} + \frac{i}{2}f_{[b}{}^c(\bar{\psi}_{a]}{}^j\bar{\sigma}_c)^{\alpha} + \frac{1}{16}\nabla_f W_{dc}^-(\psi_{[b}{}^j\bar{\sigma}_{a]}{}^{\alpha}\bar{\sigma}_{dc}\bar{\sigma}_f)^{\alpha} \\ &\quad + \frac{i}{8}R(I)_{dc}{}_k{}^j(\bar{\psi}_{[b}{}^k\bar{\sigma}_{a]}{}^{\alpha}) + \frac{1}{4}R(\mathbb{A})_{dc}(\bar{\psi}_{[b}{}^j\bar{\sigma}_{a]}{}^{\alpha}\sigma^{dc})^{\alpha} - \frac{3i}{16}D(\bar{\psi}_{[b}{}^j\bar{\sigma}_{a]}{}^{\alpha})^{\alpha} \\ &\quad + \frac{3}{8}(\bar{\psi}_{[b}\bar{\psi}_{a]})\chi^{\alpha j}. \end{aligned} \quad (3.71)$$

It should be emphasized that the above expression results from the choice $a = -2$.

The constraints imposed by superspace are naturally written in spinor form. We find first

$$\begin{aligned} R(K)_{\dot{\gamma}\dot{\beta}}{}^{\alpha i} &= -\frac{i}{16}\nabla_{\alpha}\dot{\phi}\bar{\nabla}_{\dot{\gamma}}{}^j\bar{W}_{\dot{\beta}\dot{\phi}}| - \frac{i}{16}\nabla_{\alpha}\dot{\phi}\bar{\nabla}_{\dot{\beta}}{}^j\bar{W}_{\dot{\gamma}\dot{\phi}}| - \frac{1}{8}\bar{W}_{\dot{\gamma}\dot{\beta}}\nabla^{\phi j}W_{\phi\alpha}| \\ &= \frac{i}{4}\nabla_{\alpha}\dot{\phi}T_{\dot{\phi}\dot{\gamma}\dot{\beta}}{}^j + \frac{i}{4}\nabla_{\alpha}\dot{\phi}T_{\dot{\phi}\dot{\beta}\dot{\gamma}}{}^j + \frac{3}{8}\bar{W}_{\dot{\gamma}\dot{\beta}}\chi_{\alpha}{}^j. \end{aligned} \quad (3.72)$$

This constraint is actually obeyed identically. It corresponds to part of the spinor decomposition of eq. (A.13) of [13]. The other part of the curvature tensor is

$$R(K)_{\gamma\beta}{}^{\alpha i} = \frac{1}{96}(\nabla^{ij}\nabla_{\gamma j}W_{\beta\alpha}| + \nabla^{ij}\nabla_{\beta j}W_{\gamma\alpha}|). \quad (3.73)$$

Here one must recall that $W_{\alpha\beta}$ is constrained by its Bianchi identity (2.36). In particular, the Bianchi identity implies

$$\nabla^{ij}\nabla_{\gamma j}W_{\gamma\alpha} = -6i\nabla_{\alpha\dot{\beta}}\bar{\nabla}_{\dot{\gamma}}{}^i\bar{W}^{\dot{\gamma}\dot{\beta}} \quad (3.74)$$

so the spin-1/2 part of (3.73) is constrained; the spin-3/2 part serves to define the totally symmetric $\nabla^{ij}\nabla_{(\gamma j}W_{\beta\alpha)}|$. Note that this combination is the last remaining component of the $W_{\alpha\beta}$ which requires definition; all other components have been fixed or are related to already defined objects by the Bianchi identity.

Taking just the spin-1/2 part of the curvature, we find

$$R(S)_{\alpha\beta}{}^{\beta j} = \frac{3i}{16}\nabla_{\beta\dot{\phi}}\bar{\chi}^{\dot{\phi}j}. \quad (3.75)$$

This is a constraint, but it corresponds to the rest of the spinor decomposition of eq. (A.13) of [13]. Thus the constraints on the special conformal curvature are *exactly* as in the superconformal tensor calculus, and so they are satisfied identically.

3.8 Special conformal curvature

The special conformal curvature is given by

$$R(K)_{nm}{}^a = \mathcal{D}_{[n} f_{m]}{}^a + \frac{i}{2}(\phi_{[n} \sigma^a \bar{\phi}_{m]}). \quad (3.76)$$

Again we have suppressed isospin indices, taking

$$(\phi_n \sigma^a \bar{\phi}_m) = (\phi_n^j \sigma^a \bar{\phi}_{mj}) \quad (3.77)$$

with the indices in their natural positions.

Its supercovariant form is a bit of an exercise in restoring σ matrices everywhere but otherwise is completely straightforward. The result may be written

$$\begin{aligned} R(K)_{cb}{}^a &= e_c{}^n e_b{}^m \mathcal{D}_{[n} f_{m]}{}^a + \frac{i}{2}(\phi_{[c} \sigma^a \bar{\phi}_{b]}) + \frac{i}{2}(\psi_{[c} \sigma^a \nabla_d T^{da}{}_{b]}) + \frac{i}{2}(\bar{\psi}_{[c}{}^j \bar{\sigma}_b] \nabla_d T^{da}{}_{j}) \\ &\quad + \frac{1}{4}(\psi_c \psi_b) \nabla_d W^{-da} + \frac{1}{4}(\bar{\psi}_c \bar{\psi}_b) \nabla_d W^{+da}. \end{aligned} \quad (3.78)$$

It turns out that the special conformal curvature is completely constrained by the superspace structure to equal other objects; however, this is exactly the same situation as in STC. The precise form of the curvature in terms of superfields is given in (B.49); we focus merely on highlighting the self-dual part in spinor notation:

$$R(K)_{\beta\alpha}{}^{\gamma\dot{\gamma}} = -\frac{1}{2}\nabla_{\dot{\gamma}}{}^{\phi} R_{\phi\gamma\beta\alpha} - \sum_{\beta\alpha} \left(\frac{3i}{4} \bar{\chi}_{\dot{\gamma}j} T_{\gamma\beta\alpha}{}^j + \frac{1}{8} \nabla_{\alpha\dot{\phi}} (\bar{W}^{\dot{\phi}}{}_{\dot{\gamma}} W_{\beta\gamma}) + \frac{3}{8} \epsilon_{\alpha\gamma} \nabla_{\beta\dot{\gamma}} D \right). \quad (3.79)$$

It is a straightforward (and tedious) exercise to show that this form is implied by the relation (A.8) of [13], which is an identity on the component fields.

4 Reduction to $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_R$

Howe's formulation of $\mathcal{N} = 2$ conformal supergravity [18] involved the choice of structure group $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_R$, with the corresponding covariant derivative

$$\mathcal{D}_A = E_A{}^M \left(\partial_M - \frac{1}{2} \Omega_M{}^{ba} M_{ab} - A_M \mathbb{A} - \Phi_M{}^i{}_j I^j{}_i \right) \quad (4.1)$$

with the algebra

$$[\mathcal{D}_A, \mathcal{D}_B] = -\hat{T}_{AB}{}^C - \frac{1}{2} \hat{R}_{AB}{}^{cd} M_{dc} - \hat{R}_{AB}{}^i{}_j I^j{}_i - \hat{R}_{AB} \mathbb{A}. \quad (4.2)$$

In order for the component structure of the theory to correspond to conformal supergravity, the torsion tensor must be constrained by

$$\begin{aligned} \hat{T}_{\alpha\beta}{}^C &= \hat{T}^{\dot{\alpha}\dot{\beta}C} = \hat{T}_{\alpha}{}^{\dot{\beta}\gamma} = \hat{T}_{\alpha}{}^{\dot{\beta}}{}_{\dot{\gamma}} = 0 \\ \hat{T}_{\alpha}{}^{i\dot{\beta}}{}_{j}{}^c &= 2i \delta_j^i (\sigma^c)_{\alpha}{}^{\dot{\beta}} \\ \hat{T}_{Ab}{}^c &= 0, \quad \hat{T}_{\alpha\dot{\alpha}\beta}{}^{j\gamma}{}_k = \frac{1}{2} \delta_{\alpha}{}^{\gamma} \hat{T}_{\dot{\alpha}\beta}{}^{j\phi}{}_k \end{aligned} \quad (4.3)$$

The solution to these constraints is then uniquely given by Howe's superspace formulation.¹⁷ The superfield content of this theory involves three complex superfields¹⁸

$$W_{\alpha\beta} = W_{\beta\alpha}, \quad Y_{\alpha\beta} = Y_{\beta\alpha}, \quad S_{ij} = S_{ji}, \quad (4.4)$$

and two real superfields

$$G_a, \quad G_a{}^{ij} = G_a{}^{ji} \quad (4.5)$$

along with a set of constraints,

$$\mathcal{D}_\alpha{}^i S^{jk} = 0, \quad \bar{\mathcal{D}}_{\dot{\alpha}}{}^i S^{jk} = i\mathcal{D}^{\beta(i} G_{\beta\dot{\alpha}}{}^{jk)} \quad (4.6a)$$

$$\mathcal{D}_{(\alpha}{}^i Y_{\beta\gamma)} = 0, \quad \mathcal{D}_\alpha{}^i S_{ij} + \mathcal{D}^\beta{}_j Y_{\beta\alpha} = 0 \quad (4.6b)$$

$$\mathcal{D}_{(\alpha}{}^i G_{\beta)\dot{\beta}}{}^{jk} = 0 \quad (4.6c)$$

$$\mathcal{D}_\alpha{}^i G_{\beta\dot{\beta}} = -i\bar{\mathcal{D}}_{\dot{\beta}}{}^i Y_{\beta\alpha} + \frac{1}{12}\epsilon_{\alpha\beta}\bar{\mathcal{D}}_{\dot{\beta}j}S^{ji} - \frac{1}{4}\epsilon_{\alpha\beta}\bar{\mathcal{D}}^{\dot{\gamma}i}\bar{W}_{\dot{\gamma}\dot{\beta}} - \frac{i}{3}\epsilon_{\alpha\beta}\mathcal{D}^\gamma{}_j G_{\gamma\dot{\beta}}{}^{ji}. \quad (4.6d)$$

What makes this formulation conformal is that the torsion constraints admit a super-Weyl transformation involving a real unconstrained superfield parameter Λ . In terms of this super-Weyl transformation, $W_{\alpha\beta}$ transforms homogeneously, but all the other fields above transform inhomogeneously:

$$\delta W_{\alpha\beta} = \Lambda W_{\alpha\beta} \quad (4.7a)$$

$$\delta Y_{\alpha\beta} = \Lambda Y_{\alpha\beta} - \frac{1}{2}\mathcal{D}_{\alpha\beta}\Lambda \quad (4.7b)$$

$$\delta S_{ij} = \Lambda S_{ij} - \frac{1}{2}\mathcal{D}_{ij}\Lambda \quad (4.7c)$$

$$\delta G_{\alpha\dot{\alpha}} = \Lambda G_{\alpha\dot{\alpha}} - \frac{1}{8}[\mathcal{D}_\alpha{}^k, \bar{\mathcal{D}}_{\dot{\alpha}k}]\Lambda \quad (4.7d)$$

$$\delta G_{\alpha\dot{\alpha}}{}^{ij} = \Lambda G_{\alpha\dot{\alpha}}{}^{ij} + \frac{i}{4}[\mathcal{D}_\alpha{}^i, \bar{\mathcal{D}}_{\dot{\alpha}}{}^j]\Lambda. \quad (4.7e)$$

It is clear that $W_{\alpha\beta}$ should be the field of the same name from conformal superspace; the other fields should have an origin which sheds light on their transformation properties and their occurrence in the curvature and torsion tensors.

4.1 Conventional degauging

Our structure group differs from the group $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_\text{R}$ by the addition of the dilatation symmetry and the special conformal generators K_A . As Howe's formulation has as an extra symmetry of the constraints only the super-Weyl transformation, we must fix the special conformal symmetry in the degauging procedure.

¹⁷See [20] for a recent detailed review of Howe's formulation of $\mathcal{N} = 2$ conformal supergravity.

¹⁸It should be noted that when comparing our notation to [19, 20], we have an extra sign appearing in the Lorentz decomposition of an antisymmetric two-form (2.8). Thus while our $\bar{W}_{\dot{\alpha}\dot{\beta}}$ and $Y_{\alpha\beta}$ match those of [19, 20], our $W_{\alpha\beta}$ and $\bar{Y}_{\dot{\alpha}\dot{\beta}}$ differ by a sign.

It turns out there is an obvious way of doing this, and it works the same way for $\mathcal{N} = 2$ as for $\mathcal{N} \leq 1$. Under a special conformal transmutation with parameter ϵ^A , the dilatation gauge field B_M transforms as

$$\delta_K(\epsilon)B_M = -2\epsilon^a E_{Ma} + 2\epsilon^{\alpha j} E_{M\alpha j} + 2\epsilon_{\dot{\alpha} j} E_M^{\dot{\alpha} j}. \quad (4.8)$$

It is evidently possible that we may exhaust the K -gauge freedom via the gauge choice $B_M = 0$. This not only fixes the special conformal symmetry but also removes the dilatation connection from all covariant derivatives.

However, the covariant derivatives still carry the special conformal connection F_M^A . As the symmetry associated with this connection has been fixed, it should no longer be considered as part of the covariant derivative. We therefore rewrite

$$\nabla_A = \mathcal{D}_A - F_A^B K_B, \quad F_A^B \equiv E_A^M F_M^B \quad (4.9)$$

where \mathcal{D}_A is the $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_R$ covariant derivative (4.1).

It is quite straightforward to work out how the Poincaré curvatures are related to the conformal curvatures. For example, the torsion tensors are related by

$$T^a = \hat{T}^a, \quad T^\alpha_i = \hat{T}^\alpha_i + iE^b \wedge F_{\gamma i} \bar{\sigma}_b^{\dot{\gamma}\alpha}, \quad T_{\dot{\alpha}}^i = \hat{T}_{\dot{\alpha}}^i + iE_b \wedge F^{\gamma i} \sigma_{\gamma\dot{\alpha}}^b. \quad (4.10)$$

The difference in the terms is so simple that it is clear to see that the relations

$$\begin{aligned} T_{\alpha\beta}^C &= T^{\dot{\alpha}\dot{\beta}C} = T_{\alpha}^{\dot{\beta}\gamma} = T_{\alpha}^{\dot{\beta}}{}_{\dot{\gamma}} = 0 \\ T_{\alpha}^{\dot{\beta}}{}_{\dot{j}}{}^c &= 2i\delta_j^i (\sigma^c)_{\alpha}^{\dot{\beta}} \\ T_{Ab}{}^c &= 0, \quad T_{\alpha\dot{\alpha}\beta}{}^{j\gamma}{}_k = \frac{1}{2}\delta_{\alpha}^{\gamma} T_{\phi\dot{\alpha}\beta}{}^{j\phi}{}_k \end{aligned} \quad (4.11)$$

obeyed by the conformal torsion tensor are *also* obeyed by the Poincaré torsion tensors. These relations are precisely the torsion tensor constraints (4.3) which together with the choice of structure group uniquely determine Howe's formulation of conformal supergravity. Our structure *must* therefore correspond to Howe's solution when degauged. We will now show this explicitly by solving the constraints obeyed by the connections F_A^B .

4.2 The conformal origin of $\mathcal{N} = 2$ auxiliary superfields

In the gauge where $B_M = 0$, the conformal dilatation curvature is given by

$$R(\mathbb{D})_{BA} = +2F_{BA}(-)^a - 2F_{AB}(-)^{b+ab}. \quad (4.12)$$

The purely chiral and antichiral spinor curvatures vanish, giving

$$R(\mathbb{D})_{\beta\alpha} = 0 \implies F_{\beta\alpha} = -F_{\alpha\beta} \quad (4.13a)$$

$$R(\mathbb{D})^{\dot{\beta}\dot{\alpha}} = 0 \implies F^{\dot{\beta}\dot{\alpha}} = -F^{\dot{\alpha}\dot{\beta}}. \quad (4.13b)$$

These imply the Lorentz and isospin decompositions

$$F_{\beta}{}^j{}_{\alpha}{}^i = \frac{1}{2}\epsilon_{\beta\alpha} S^{ji} - \frac{1}{2}\epsilon^{ji} Y_{\beta\alpha} \quad (4.14a)$$

$$F_{\dot{j}}{}^{\dot{\beta}}{}_{\dot{i}}{}^{\dot{\alpha}} = \frac{1}{2}\epsilon^{\dot{\beta}\dot{\alpha}} \bar{S}_{ji} + \frac{1}{2}\epsilon_{ji} \bar{Y}^{\dot{\beta}\dot{\alpha}}. \quad (4.14b)$$

The mixed spinor curvature also vanishes, giving

$$R(\mathbb{D})_{\beta}^{\dot{\alpha}} = 0 \implies F_{\beta}^{\dot{\alpha}} = -F^{\dot{\alpha}}_{\beta}. \quad (4.15)$$

This implies the isospin decompositions

$$F_{\alpha}^{\dot{i}\dot{\alpha}j} = +\delta_j^i G_{\alpha\dot{\alpha}} + i G_{\alpha\dot{\alpha}}^{\dot{i}j} \quad (4.16a)$$

$$F^{\dot{\alpha}}_{\dot{i}\alpha j} = -\delta_i^j G^{\dot{\alpha}\alpha} - i G^{\dot{\alpha}\alpha j}_{\dot{i}} \quad (4.16b)$$

where G_c and G_c^{ij} are real isosinglet and isotriplet vectors, respectively.

In identifying these superfields as elements of the special conformal connections of the gauge-fixed geometry, we can give an alternative explanation of their super-Weyl transformation properties. Because Howe's structure is given by the choice $B_M = 0$, any dilatation must be accompanied by a special conformal transformation which restores this gauge choice. We find

$$0 = \delta_{SW}(\Lambda)B_M = \delta_D(\Lambda)B_M + \delta_K(\epsilon^A)B_M \quad (4.17)$$

for the choice

$$\epsilon^a = \frac{1}{2}\mathcal{D}^a\Lambda, \quad \epsilon^{\alpha j} = -\frac{1}{2}\mathcal{D}^{\alpha j}\Lambda, \quad \epsilon_{\dot{\alpha}j} = -\frac{1}{2}\bar{\mathcal{D}}_{\dot{\alpha}j}\Lambda. \quad (4.18)$$

The δ_K in the above expression generates the inhomogeneous part of the super-Weyl transformation. Using $\delta_K F_M^{\beta j} = \mathcal{D}_M \epsilon^{\beta j} - i E_{M\beta}^j \epsilon^{\dot{\beta}\beta}$ and its conjugate, we find

$$\delta_K S^{ij} = \delta_K F^{\alpha i}_{\alpha}{}^j = \mathcal{D}^{\alpha i} \epsilon_{\alpha}{}^j = -\frac{1}{2}\mathcal{D}^{ij}\Lambda \quad (4.19a)$$

$$\delta_K Y_{\alpha\beta} = \delta_K F_{\beta}^k{}_{\alpha k} = -\frac{1}{2}\mathcal{D}_{\beta\alpha}\Lambda \quad (4.19b)$$

$$\delta_K G_{\alpha\dot{\alpha}} = \frac{1}{2}\delta_K F_{\alpha}^k{}_{\dot{\alpha}k} = \frac{1}{2}\mathcal{D}_{\alpha}{}^k \epsilon_{\dot{\alpha}k} - i \epsilon_{\alpha\dot{\alpha}} = -\frac{1}{8}[\mathcal{D}_{\alpha}{}^k, \bar{\mathcal{D}}_{\dot{\alpha}k}]\Lambda \quad (4.19c)$$

$$\delta_K G_{\alpha\dot{\alpha}}^{ij} = -i\delta_K F_{\alpha}^{(i}{}_{\dot{\alpha}}{}^{j)} = \frac{i}{4}[\mathcal{D}_{\alpha}^{(i}, \mathcal{D}_{\dot{\alpha}}^{j)}]\Lambda \quad (4.19d)$$

which are precisely the inhomogeneous parts of the transformation laws for these objects as given in (4.7).

4.3 The conformal origin of constraints

The superfields of Howe's superspace formulation are constrained by a set of dimension-3/2 Bianchi identities. It turns out that these relations are encoded in the structure of conformal superspace in the constraints on the dimension-3/2 dilatation and conformal curvatures.

We begin by considering the dimension-3/2 dilatation curvature. In the gauge where $B_M = 0$, we have

$$R(\mathbb{D})_{\beta}^j{}_{\alpha\dot{\alpha}} = -\frac{i}{2}\epsilon_{\beta\alpha}\bar{\nabla}_{\dot{\phi}}{}^j\bar{W}^{\dot{\phi}}{}_{\dot{\alpha}} = 2F_{\beta}^j{}_{\alpha\dot{\alpha}} + 2F_{\alpha\dot{\alpha}}{}_{\beta}{}^j \quad (4.20)$$

and similarly for the conjugate expression, yielding

$$F_{\alpha\dot{\alpha}\beta}^j + F_{\beta}^j{}_{\alpha\dot{\alpha}} = -\frac{i}{4}\epsilon_{\beta\alpha}\bar{\nabla}_{\dot{\phi}}^j\bar{W}^{\dot{\phi}}{}_{\dot{\alpha}} \quad (4.21a)$$

$$F_{\alpha\dot{\alpha}\dot{\beta}j} + F_{\dot{\beta}j\alpha\dot{\alpha}} = -\frac{i}{4}\epsilon_{\dot{\beta}\dot{\alpha}}\nabla_j^{\phi}W_{\phi\alpha}. \quad (4.21b)$$

Next, we need to consider the dimension-3/2 components of the special conformal curvatures. In general, we have¹⁹

$$R(K)_{CB}{}^a = \mathcal{D}_{[C}F_{B]}{}^a + \hat{T}_{CB}{}^D F_D{}^a + 2iF_{[C}{}^{\alpha j}F_{B]}{}_{\dot{\alpha}j}\sigma_{\alpha\dot{\alpha}}^a \quad (4.22a)$$

$$R(K)_{CB}{}^{\alpha i} = \mathcal{D}_{[C}F_{B]}{}^{\alpha i} + \hat{T}_{CB}{}^D F_D{}^{\alpha i} - i\delta_{[C}{}^i F_{B]}{}^a \bar{\sigma}_a{}^{\dot{\alpha}\alpha} \quad (4.22b)$$

$$R(K)_{CB\dot{\alpha}i} = \mathcal{D}_{[C}F_{B]}{}_{\dot{\alpha}i} + \hat{T}_{CB}{}^D F_{D\dot{\alpha}i} - i\delta_{[C}{}^{\alpha} F_{B]a}\sigma_{\alpha\dot{\alpha}}^a \quad (4.22c)$$

where $\hat{T}_{CB}{}^D$ is the Poincaré torsion tensor. There are three independent dimension-3/2 constraints among the special conformal curvatures. The simplest is

$$0 = R_{\beta\alpha}{}^{\gamma k} \equiv \mathcal{D}_{\beta}F_{\alpha}{}^{\gamma k} + \mathcal{D}_{\alpha}F_{\beta}{}^{\gamma k}. \quad (4.23)$$

Plugging in the form of $F_{\beta}^j{}_{\gamma}{}^k$, we find

$$\mathcal{D}_{\beta}{}^{(k}S^{ij)} = 0, \quad \mathcal{D}_{(\gamma}{}^jY_{\beta\alpha)} = 0, \quad \mathcal{D}_{\alpha j}S^{ji} = \mathcal{D}^{\beta i}Y_{\beta\alpha}. \quad (4.24)$$

The second constraint is

$$2\epsilon_{\beta\alpha}\epsilon^{ji}\mathcal{W}(S)_{\dot{\gamma}k} = R_{\beta}^j{}_{\alpha}{}^i{}_{\dot{\gamma}k} \equiv \mathcal{D}_{\beta}F_{\alpha}{}^i{}_{\dot{\gamma}k} + \mathcal{D}_{\alpha}F_{\beta}{}^i{}_{\dot{\gamma}k} + i\delta_k^j F_{\alpha}{}^i{}_{\beta\dot{\alpha}} + i\delta_k^i F_{\beta}{}^j{}_{\alpha\dot{\alpha}}. \quad (4.25)$$

Inserting the form of $F_{\alpha}{}^i{}_{\dot{\gamma}k}$ yields the solution

$$iF_{\beta}^j{}_{\alpha\dot{\alpha}} = -\mathcal{D}_{\beta}{}^jG_{\alpha\dot{\alpha}} - \frac{i}{3}\mathcal{D}_{(\beta}{}^kG_{\alpha)\dot{\alpha}}{}^j{}_k + \frac{i}{2}\epsilon_{\beta\alpha}\mathcal{D}^{\phi k}G_{\phi\dot{\alpha}}{}^j{}_k + 2\epsilon_{\beta\alpha}\bar{\mathcal{W}}(S)_{\dot{\alpha}}{}^j \quad (4.26)$$

along with the additional constraint

$$\mathcal{D}_{(\beta}{}^{(k}G_{\alpha)\dot{\alpha}}{}^{ij)} = 0. \quad (4.27)$$

Finally, the constraint for $R(S)_{\beta}{}^{\dot{\alpha}\gamma k}$ gives

$$0 = R(S)_{\beta}{}^{\dot{\alpha}i}{}_{\gamma}{}^k = \mathcal{D}_{\beta}{}^jF_{\dot{\alpha}i}{}_{\gamma}{}^k + \bar{\mathcal{D}}_{\dot{\alpha}i}F_{\beta}{}^j{}_{\gamma}{}^k + 2i\delta_i^j F_{\beta\dot{\alpha}}{}_{\gamma}{}^k + i\delta_i^k F_{\beta}{}^j{}_{\gamma\dot{\alpha}}. \quad (4.28)$$

Plugging in the form for each of these superfields leads to

$$\bar{\mathcal{D}}_{\dot{\alpha}}{}^{(i}S^{jk)} = i\mathcal{D}^{\phi(i}G_{\phi\dot{\alpha}}{}^{jk)} \quad (4.29)$$

$$\mathcal{D}_{\beta}{}^jG_{\alpha\dot{\alpha}} = -\frac{1}{4}\bar{\mathcal{D}}_{\dot{\alpha}}{}^jY_{\beta\alpha} + \frac{1}{12}\epsilon_{\beta\alpha}\bar{\mathcal{D}}_{\dot{\alpha}k}S^{kj} + \frac{i}{3}\epsilon_{\beta\alpha}\mathcal{D}^{\phi k}G_{\phi\dot{\alpha}}{}^j{}_k + \frac{1}{4}\epsilon_{\beta\alpha}\bar{\mathcal{D}}_{\dot{\phi}}{}^j\bar{W}^{\dot{\phi}}{}_{\dot{\alpha}}. \quad (4.30)$$

This reproduces all of the dimension-3/2 constraints in Howe's superspace geometry [18, 20].

In the interest of fully expressing the degauged parameters, we should also mention how to solve for $F_b{}^a$, which is a dimension two superfield. It is easily found in the constraint for $R(K)_{\gamma}{}^{\dot{\beta}a}$

$$0 = R(K)_{\gamma}{}^{\dot{\beta}j}{}_{\alpha\dot{\alpha}} = \mathcal{D}_{\gamma}{}^kF_{\dot{\beta}j\alpha\dot{\alpha}} + \bar{\mathcal{D}}_{\dot{\beta}j}F_{\gamma}{}^k{}_{\alpha\dot{\alpha}} + 2i\delta_j^k F_{\gamma\dot{\beta}}{}_{\alpha\dot{\alpha}} + 4iF_{\gamma}{}^k{}_{\alpha}{}^l F_{\dot{\beta}j\dot{\alpha}l} + 4iF_{\dot{\beta}j\dot{\alpha}}{}^l F_{\gamma}{}^k{}_{\dot{\alpha}l} \quad (4.31)$$

in terms of quantities already defined.

¹⁹The gradings have been suppressed. In the first term of each expression, $[CB]$ means $CB - BC(-)^{bc}$ whereas in each of the third terms, $[CB]$ means $(-)^b CB - BC(-)^{bc+c}$.

4.4 Reproducing degauged curvatures

It is a straightforward task to reproduce the curvatures of the $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_\text{R}$ geometry via this same degauging process. The most straightforward way is to take the curvature two-forms in the conformal geometry and write them as the degauged curvatures. For example, suppose we want to calculate the components of the isospin curvature tensor $R(I)_{\beta\alpha kl}$. Using its explicit form in terms of the superconnections (B.33), we may identify

$$R(I)_{\beta\alpha}{}^j{}_i{}^k{}_l = \hat{R}(I)_{\beta\alpha}{}^j{}_i{}^k{}_l + 4\delta_{(k}^j F_{\alpha}{}^i{}_{\beta l)} + 4\delta_{(k}^i F_{\beta}{}^j{}_{\alpha l)}. \quad (4.32)$$

Because this curvature is constrained to be zero, we may identify the degauged curvature

$$\hat{R}(I)_{\beta\alpha}{}^j{}_i{}^k{}_l = -4\delta_{(k}^j F_{\alpha}{}^i{}_{\beta l)} - 4\delta_{(k}^i F_{\beta}{}^j{}_{\alpha l)} = -4\delta_{(k}^j \delta_{l)}^i Y_{\alpha\beta}. \quad (4.33)$$

An equivalent (but more efficient) way of proceeding is to calculate the desired commutators directly. For example, to calculate $\{\mathcal{D}_\alpha{}^i, \mathcal{D}_\beta{}^j\}$, one may consider a conformally primary field Ψ lying in some arbitrary representation of the rest of the superconformal algebra. Then observing that

$$\nabla_\alpha = \mathcal{D}_\alpha - F_\alpha{}^C K_C \quad (4.34)$$

we may calculate

$$\begin{aligned} \nabla_\alpha \nabla_\beta \Psi &= \mathcal{D}_\alpha \nabla_\beta \Psi - F_\alpha{}^C [K_C, \nabla_\beta] \Psi \\ &= \mathcal{D}_\alpha \mathcal{D}_\beta \Psi - F_\alpha{}^C [K_C, \nabla_\beta] \Psi. \end{aligned} \quad (4.35)$$

Symmetrizing the indices and reordering the expression, we are led to

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} \Psi = \{\nabla_\alpha, \nabla_\beta\} \Psi + F_\alpha{}^C [K_C, \nabla_\beta] \Psi + F_\beta{}^C [K_C, \nabla_\alpha] \Psi. \quad (4.36)$$

Since the only special conformal generator in the commutator which will yield a non-vanishing result is S , we have

$$\{\mathcal{D}_\alpha{}^i, \mathcal{D}_\beta{}^j\} \Psi = -\epsilon_{\alpha\beta} \epsilon^{ij} \bar{W}_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta}\dot{\alpha}} \Psi + F_\alpha{}^{i\gamma k} \{S_{\gamma k}, \nabla_\beta{}^j\} \Psi + F_\beta{}^{j\gamma k} \{S_{\gamma k}, \nabla_\alpha{}^i\} \Psi. \quad (4.37)$$

Evaluating the commutators and inserting the forms for F gives

$$\begin{aligned} \{\mathcal{D}_\alpha{}^i, \mathcal{D}_\beta{}^j\} &= -\epsilon_{\alpha\beta} \epsilon^{ij} \bar{W}_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta}\dot{\alpha}} + 2S^{ij} M_{\alpha\beta} + \epsilon^{ij} \epsilon_{\alpha\beta} Y^{\delta\phi} M_{\phi\delta} \\ &\quad + 2\epsilon_{\alpha\beta} \epsilon^{ij} S^k{}_l I^l{}_k - 4Y_{\alpha\beta} I^{ij}. \end{aligned} \quad (4.38)$$

All of the commutators may be evaluated in either way.

4.5 Action principle for $\text{SL}(2, \mathbb{C}) \times \text{U}(2)_\text{R}$ supergravity

In the previous section we have given the form for the component evaluation of a chiral action as well as the rule for transforming a full superspace action into a chiral one. These are well-known procedures in $\text{U}(2)_\text{R}$ supergravity; since we may degauge our formulae to regular $\text{U}(2)_\text{R}$ supergravity, it is a straightforward exercise to verify that we reproduce these results.

Recall that a full conformal superspace action may be converted to a chiral action in a very simple way:

$$\int d^{12}z E \mathcal{L} = \int d^8z \mathcal{E} \bar{\nabla}^4 \mathcal{L}, \quad \bar{\nabla}^4 \equiv \frac{1}{48} \bar{\nabla}^{ij} \bar{\nabla}_{ij}. \quad (4.39)$$

Because of the simplicity of the conformal superspace curvatures, it is simple to show that we could just as easily have written

$$\bar{\nabla}^4 \mathcal{L} = \frac{1}{96} \bar{\nabla}^{ij} \bar{\nabla}_{ij} \mathcal{L} - \frac{1}{96} \bar{\nabla}^{\dot{\alpha}\dot{\beta}} \bar{\nabla}_{\dot{\alpha}\dot{\beta}} \mathcal{L} \quad (4.40)$$

since the two terms are equivalent in conformal supergravity when \mathcal{L} is real with vanishing conformal weight. We would like to degauge both of them. This is most easily done working from the outside in. Keeping in mind that \mathcal{L} has vanishing conformal and $U(1)_R$ weights, we find for the first term

$$\begin{aligned} \bar{\nabla}^{ij} \bar{\nabla}_{ij} \mathcal{L} &= \bar{\mathcal{D}}_{\dot{\alpha}}^i \bar{\nabla}^{\dot{\alpha}j} \bar{\nabla}_{ij} \mathcal{L} + 6 \bar{S}_{ij} \bar{\nabla}^{ij} \mathcal{L} + 6 \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\nabla}^{\dot{\beta}\dot{\alpha}} \mathcal{L} \\ &= \bar{\mathcal{D}}_{\dot{\alpha}}^i (\bar{\mathcal{D}}^{\dot{\alpha}j} \bar{\nabla}_{ij} \mathcal{L} - 2 \bar{S}_{ij} \bar{\nabla}^{\dot{\alpha}j} \mathcal{L} + 6 \bar{Y}^{\dot{\alpha}\dot{\beta}} \bar{\nabla}_{\dot{\beta}i} \mathcal{L}) + 6 \bar{S}_{ij} \bar{\nabla}^{ij} \mathcal{L} + 6 \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\nabla}^{\dot{\beta}\dot{\alpha}} \mathcal{L}. \end{aligned}$$

Observing that $\bar{\nabla}^{\dot{\alpha}}_i \mathcal{L} = \mathcal{D}^{\dot{\alpha}}_i \mathcal{L}$ and $\bar{\nabla}_{ij} \mathcal{L} = \bar{\mathcal{D}}_{ij} \mathcal{L}$, we find

$$\bar{\nabla}^{ij} \bar{\nabla}_{ij} \mathcal{L} = \bar{\mathcal{D}}_{ij} \bar{\mathcal{D}}^{ij} \mathcal{L} + 4 \bar{S}_{ij} \bar{\mathcal{D}}^{ij} \mathcal{L} + 12 \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} \mathcal{L} - 2 \mathcal{D}_{\dot{\alpha}}^i \bar{S}_{ij} \bar{\mathcal{D}}^{\dot{\alpha}j} \mathcal{L} + 6 \bar{\mathcal{D}}_{\dot{\alpha}}^j \bar{Y}^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}_{\dot{\beta}j} \mathcal{L}.$$

Similarly we may calculate

$$\bar{\nabla}_{\dot{\alpha}\dot{\beta}} \bar{\nabla}^{\dot{\alpha}\dot{\beta}} \mathcal{L} = \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} \mathcal{L} - 4 \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\beta}\dot{\alpha}} \mathcal{L} - 12 \bar{S}_{ij} \bar{\mathcal{D}}^{ij} \mathcal{L} - 6 \bar{\mathcal{D}}_{\dot{\alpha}}^j \bar{S}_{jk} \bar{\mathcal{D}}^{\dot{\alpha}k} \mathcal{L} + 2 \bar{\mathcal{D}}_{\dot{\alpha}}^j \bar{Y}^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}_{\dot{\beta}j} \mathcal{L}.$$

Putting these together, we find

$$\bar{\nabla}^4 \mathcal{L} = \frac{1}{96} (\bar{\mathcal{D}}_{ij} \bar{\mathcal{D}}^{ij} - \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}}) \mathcal{L} + \frac{1}{6} \bar{S}_{ij} \bar{\mathcal{D}}^{ij} \mathcal{L} + \frac{1}{6} \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} \mathcal{L} \quad (4.41)$$

which is precisely the $\mathcal{N} = 2$ chiral projection operator as given in [29].

By a very nearly identical line of attack, one may degauge the component action (2.69) to reproduce the component action originally calculated by Müller [31]. We leave this as an exercise to the interested reader. (See [36] for an alternative calculation of this same component Lagrangian in notation more closely related to our own.)

5 Future directions and outlook

We have presented a single framework for $\mathcal{N} = 2$ supergravity which is simultaneously manifestly superconformal and manifestly supersymmetric. It reduces in components to the manifestly superconformal framework of de Wit et al. [5–10] and can be degauged in superspace to the manifestly supersymmetric framework constructed by Howe [18]. At the same time, it sheds light on certain curious features of both these formulations.

It is a very pleasant surprise that the basic principles of our earlier work on $\mathcal{N} = 1$ conformal superspace [21] can so easily be applied to $\mathcal{N} = 2$. This raises two very interesting

questions. First, can we extend this formulation somehow to $\mathcal{N} = 4$? Second, can we extend this to different dimensions? One very nice feature of this superconformal approach is that it can so readily reproduce the geometric structure of Poincaré supergravity. For the cases of $\mathcal{N} = 1, 2$ in four dimensions which we have explored, this reproduction isn't terribly impressive since it tells us nothing new, merely reconfirming old results in a new light. However, for other dimensions where the superspace structure of supergravity remains something of a mystery, it is a distinct possibility that superconformal procedures may offer up superspace secrets more quickly and more readily than conventional approaches.

Even within the context of $\mathcal{N} = 2$ theories in four dimensions, there remains a great deal to understand. There has been a recent flurry of activity in the supergravity structure of projective superspace, in particular its superconformal properties [19, 20, 37]. It seems very likely that the manifestly superconformal framework constructed here may offer insight. We have already begun to investigate projective superspace with a superconformal structure group and have found some promising early results. For example, the constraints (2.25) can be naturally interpreted as integrability conditions for the existence of covariantly analytic superfields:

$$\{\nabla_\alpha^+, \nabla_\beta^+\} = \{\nabla_\alpha^+, \bar{\nabla}_{\dot{\beta}}^+\} = 0, \quad \nabla_\alpha^+ \equiv v_i \nabla_\alpha^i, \quad \bar{\nabla}_{\dot{\alpha}}^+ \equiv v_i \bar{\nabla}_{\dot{\alpha}}^i. \quad (5.1)$$

Similarly, it becomes natural to interpret I_j^i of the superconformal algebra as a differential operator on the isotwistors v^i .

It would also be very interesting to understand how this superconformal approach works in harmonic superspace, where superconformal properties are less straightforwardly realized than in projective superspace.²⁰

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²⁰See for example the discussion in Chapter 9 of [28].

A Solution to the $\mathcal{N} = 2$ Bianchi identities

The constraints chosen for $\mathcal{N} = 2$ conformal supergravity in superspace are a subset of a set of constraints identical in form to super Yang-Mills in flat superspace [38]. The constraints amount to the specification of the spinor derivative anticommutators:

$$\begin{aligned}\{\nabla_\alpha^i, \nabla_\beta^j\} &= -2\epsilon_{\alpha\beta}\epsilon^{ij}\bar{\mathcal{W}}, \quad \{\bar{\nabla}_{\dot{\alpha}i}, \bar{\nabla}_{\dot{\beta}j}\} = +2\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{ij}\mathcal{W} \\ \{\nabla_\alpha^i, \bar{\nabla}_{\dot{\beta}j}\} &= -2i\delta_j^i\nabla_{\alpha\dot{\beta}}\end{aligned}\tag{A.1}$$

where \mathcal{W} and $\bar{\mathcal{W}}$ are (for the moment) arbitrary operators related by complex conjugation. In order for this structure to be consistent, we must examine all the Bianchi identities:

$$0 = \sum_{[ABC]} [\nabla_A, [\nabla_B, \nabla_C]]$$

where the sum is over graded cyclic permutations of the indices.

There are two independent dimension-3/2 Bianchi identities. The first is

$$\begin{aligned}0 &= [\nabla_\alpha^i, \{\nabla_\beta^j, \nabla_\gamma^k\}] + [\nabla_\beta^j, \{\nabla_\gamma^k, \nabla_\alpha^i\}] + [\nabla_\gamma^k, \{\nabla_\alpha^i, \nabla_\beta^j\}] \\ &= -2\epsilon_{\beta\gamma}\epsilon^{jk}[\nabla_\alpha^i, \bar{\mathcal{W}}] - 2\epsilon_{\gamma\alpha}\epsilon^{ki}[\nabla_\beta^j, \bar{\mathcal{W}}] - 2\epsilon_{\alpha\beta}\epsilon^{ij}[\nabla_\gamma^k, \bar{\mathcal{W}}].\end{aligned}$$

The constraint is satisfied provided

$$0 = [\nabla_\alpha^i, \bar{\mathcal{W}}].\tag{A.2}$$

The second dimension-3/2 Bianchi identity is

$$\begin{aligned}0 &= [\nabla_\alpha^i, \{\nabla_\beta^j, \bar{\nabla}_{\dot{\gamma}k}\}] + [\nabla_\beta^j, \{\bar{\nabla}_{\dot{\gamma}k}, \nabla_\alpha^i\}] + [\bar{\nabla}_{\dot{\gamma}k}, \{\nabla_\alpha^i, \nabla_\beta^j\}] \\ &= -2i\delta_k^j[\nabla_\alpha^i, \nabla_{\beta\dot{\gamma}}] - 2i\delta_k^i[\nabla_\beta^j, \nabla_{\alpha\dot{\gamma}}] - 2\epsilon_{\alpha\beta}\epsilon^{ij}[\bar{\nabla}_{\dot{\gamma}k}, \bar{\mathcal{W}}].\end{aligned}$$

Symmetrizing α and β , one may easily see that

$$[\nabla_\alpha^i, \nabla_{\beta\dot{\beta}}] = -2\epsilon_{\alpha\beta}\bar{\mathcal{W}}_{\dot{\beta}}^i\tag{A.3}$$

for some operator $\bar{\mathcal{W}}_{\dot{\beta}}^i$. Reinserting this result into the original expression then yields

$$\bar{\mathcal{W}}_{\dot{\beta}}^i = -\frac{i}{2}[\bar{\nabla}_{\dot{\beta}}^i, \bar{\mathcal{W}}].\tag{A.4}$$

The other dimension-3/2 Bianchi identities are related to these by complex conjugation,

$$[\bar{\nabla}_{\dot{\alpha}i}, \nabla_{\beta\dot{\beta}}] = -2\epsilon_{\alpha\beta}\mathcal{W}_{\beta i}, \quad \mathcal{W}_{\beta i} = -\frac{i}{2}[\nabla_{\beta i}, \mathcal{W}].\tag{A.5}$$

There are two independent dimension-2 Bianchi identities. The first is

$$0 = \{\nabla_\alpha^i, [\nabla_\beta^j, \nabla_{\gamma\dot{\gamma}}]\} + [\nabla_{\gamma\dot{\gamma}}, \{\nabla_\alpha^i, \nabla_\beta^j\}] - \{\nabla_\beta^j, [\nabla_{\gamma\dot{\gamma}}, \nabla_\alpha^i]\}$$

and it is satisfied automatically given the prior constraints. The second is

$$\begin{aligned} 0 &= \{\nabla_\alpha^i, [\bar{\nabla}_{\dot{\beta}j}, \nabla_{\gamma\dot{\gamma}}]\} + [\nabla_{\gamma\dot{\gamma}}, \{\nabla_\alpha^i, \bar{\nabla}_{\dot{\beta}j}\}] - \{\bar{\nabla}_{\dot{\beta}j}, [\nabla_{\gamma\dot{\gamma}}, \nabla_\alpha^i]\} \\ &= i\epsilon_{\dot{\beta}\dot{\gamma}}\{\nabla_\alpha^i, [\nabla_{\gamma j}, W]\} - 2i\delta_j^i[\nabla_{\gamma\dot{\gamma}}, \nabla_{\alpha\dot{\beta}}] + i\epsilon_{\alpha\gamma}\{\bar{\nabla}_{\dot{\beta}j}, [\bar{\nabla}_{\dot{\gamma}}^i, \bar{W}]\} \end{aligned}$$

By contracting all the spinor indices, one may show that

$$\{\nabla^{\phi i}, [\nabla_{\phi j}, \mathcal{W}]\} = \{\bar{\nabla}_{\dot{\phi}j}, [\bar{\nabla}^{\dot{\phi}i}, \bar{\mathcal{W}}]\}. \quad (\text{A.6})$$

Making use of this result in the prior expression, one finds

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = -\frac{1}{4}\epsilon_{\dot{\alpha}\dot{\beta}}\{\nabla_{(\alpha}{}^k, [\nabla_{\beta)k}, \mathcal{W}]\} + \frac{1}{4}\epsilon_{\alpha\beta}\{\bar{\nabla}_{(\dot{\alpha}}{}^k, [\bar{\nabla}_{\dot{\beta})k}, \bar{\mathcal{W}}]\}. \quad (\text{A.7})$$

The remaining dimension-5/2 and dimension-3 Bianchi identities lead to no new results.

B Curvatures of conformal superspace

In this section, we summarize the results for the conformal curvatures. The conformal covariant derivative is given by

$$\nabla_A = E_A{}^M \left(\partial_M - \frac{1}{2}\Omega_M{}^{cb}M_{bc} - \Phi_M{}^i{}_j I^j{}_i - A_M \mathbb{A} - B_M \mathbb{D} - F_M{}^B K_B \right) \quad (\text{B.1})$$

and its curvatures are

$$[\nabla_A, \nabla_B] = -T_{AB}{}^C \nabla_C - \frac{1}{2}R_{AB}{}^{cd}M_{dc} - R_{AB}{}^i{}_j I^j{}_i - R(\mathbb{A})_{AB}\mathbb{A} - R(\mathbb{D})_{AB}\mathbb{D} - R_{AB}{}^C K_C. \quad (\text{B.2})$$

We summarize the results for each curvature.

B.1 Torsion

The torsion two-forms are defined in terms of the gauge connections as

$$T^a = dE^a + E^b \wedge \Omega_b{}^a + E^a \wedge B \quad (\text{B.3})$$

$$T^\alpha{}_i = dE^\alpha{}_i + \frac{1}{2}E^\alpha{}_i \wedge B - iE^\alpha{}_i \wedge A + E^\beta{}_i \wedge \Omega_\beta{}^\alpha + E^\alpha{}_j \wedge \Phi^j{}_i + iE^b \wedge F_{\dot{\gamma}i} \bar{\sigma}_b{}^{\dot{\gamma}\alpha} \quad (\text{B.4})$$

$$T_{\dot{\alpha}}{}^i = dE_{\dot{\alpha}}{}^i + \frac{1}{2}E_{\dot{\alpha}}{}^i \wedge B + iE_{\dot{\alpha}}{}^i \wedge A + E_{\dot{\beta}}{}^i \wedge \Omega_{\dot{\alpha}}{}^{\dot{\beta}} - E_{\dot{\alpha}}{}^j \wedge \Phi_j{}^i + iE_b \wedge F^{\dot{\gamma}i} \sigma_{\dot{\gamma}\dot{\alpha}}{}^b \quad (\text{B.5})$$

The components $T_{\underline{\gamma}\underline{\beta}}{}^A$ are given by

$$T_{\underline{\gamma}\underline{\beta}}{}^{\underline{\alpha}} = 0 \quad (\text{B.6})$$

$$T_{\gamma\beta}{}^a = T^{\dot{\gamma}\dot{\beta}a} = 0, \quad T_\gamma{}^{i\dot{\beta}}{}_j{}^a = -2i\delta_j^i(\sigma^a\epsilon)_\gamma{}^{\dot{\beta}} \quad (\text{B.7})$$

The components $T_{\underline{a}b}{}^A = -\frac{1}{2}\bar{\sigma}_b^{\dot{\beta}\beta}T_{\underline{a}(\beta\dot{\beta})}{}^A$ are given by

$$T_{\underline{a}b}{}^a = 0 \quad (\text{B.8})$$

$$T_{\gamma b}{}^\alpha = T^{\dot{\gamma}}_{b\dot{\alpha}} = 0 \quad (\text{B.9})$$

$$T_{\gamma b\dot{\alpha}} \rightsquigarrow T_{\gamma}{}^j{}_{\beta\dot{\beta}}{}_{\dot{\alpha}}{}^i = i\epsilon_{\gamma\beta}\epsilon^{ji}\bar{W}_{\dot{\beta}\dot{\alpha}} \quad (\text{B.10})$$

$$T^{\dot{\gamma}}_{b\dot{\alpha}} \rightsquigarrow T^{\dot{\gamma}}{}^j{}_{\beta\dot{\beta}}{}_{\dot{\alpha}}{}^i = -i\epsilon_{\dot{\gamma}\dot{\beta}}\epsilon^{ji}W_{\beta\alpha} \quad (\text{B.11})$$

The components $T_{cb}{}^A = \frac{1}{4}\bar{\sigma}_c^{\dot{\gamma}\gamma}\bar{\sigma}_b^{\dot{\beta}\beta}T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}^A$ are given by

$$T_{cb}{}^a = 0 \quad (\text{B.12})$$

$$T_{cb}{}^\alpha \rightsquigarrow T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}_{\alpha}{}^i = \frac{1}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla_\alpha{}^i W_{\gamma\beta} \quad (\text{B.13})$$

$$T_{cb\dot{\alpha}} \rightsquigarrow T_{\gamma\dot{\gamma}\beta\dot{\beta}}{}_{\dot{\alpha}}{}^i = \frac{1}{2}\epsilon_{\gamma\beta}\bar{\nabla}_{\dot{\alpha}}{}^i \bar{W}_{\dot{\gamma}\dot{\beta}} \quad (\text{B.14})$$

B.2 Lorentz curvature

The conformal Lorentz curvature two-form is

$$R^{ba} = d\Omega^{ba} + \Omega^{bc} \wedge \Omega_c{}^a - 2E^{[b} \wedge F^{a]} - 4E^\beta{}_i \wedge F^{\alpha i} (\sigma^{ba}\epsilon)_{\alpha\beta} - 4E_{\dot{\beta}}{}^i \wedge F_{\dot{\alpha}i} (\bar{\sigma}^{ba}\epsilon)^{\dot{\alpha}\dot{\beta}} \quad (\text{B.15})$$

and may be canonically decomposed

$$R_{DC}{}^{ba} \rightsquigarrow R_{DC\beta\dot{\beta}\alpha\dot{\alpha}} = 2\epsilon_{\dot{\beta}\dot{\alpha}}R_{DC\beta\alpha} - 2\epsilon_{\beta\alpha}R_{DC\dot{\beta}\dot{\alpha}}. \quad (\text{B.16})$$

It is simplest to express the curvature results in terms of these components. We group them by dimension.

- Dimension 1

$$R_{\delta\gamma\beta\alpha} = 0, \quad R_{\delta\gamma\dot{\beta}\dot{\alpha}} = 2C_{\delta\gamma}\bar{W}_{\dot{\beta}\dot{\alpha}} \quad (\text{B.17})$$

$$R_{\delta\dot{\gamma}\dot{\beta}\dot{\alpha}} = 0, \quad R^{\dot{\delta}\dot{\gamma}}_{\beta\alpha} = -2C^{\dot{\delta}\dot{\gamma}}W_{\beta\alpha} \quad (\text{B.18})$$

$$R_{\delta}{}^{\dot{\gamma}}{}_{\beta\alpha} = R_{\delta}{}^{\dot{\gamma}}{}_{\dot{\beta}\dot{\alpha}} = 0 \quad (\text{B.19})$$

- Dimension 3/2

$$R_{\delta}{}^j{}_{\gamma\dot{\gamma}}{}_{\beta\alpha} = 0, \quad R_{\delta}{}^j{}_{\gamma\dot{\gamma}}{}_{\dot{\beta}\dot{\alpha}} = -\frac{i}{2}\epsilon_{\delta\gamma}\bar{\nabla}_{\dot{\beta}}{}^j\bar{W}_{\dot{\alpha}\dot{\gamma}} - \frac{i}{2}\epsilon_{\delta\gamma}\bar{\nabla}_{\dot{\alpha}}{}^j\bar{W}_{\dot{\beta}\dot{\gamma}} \quad (\text{B.20})$$

$$R_{\delta}{}^j{}_{\gamma\dot{\gamma}}{}_{\dot{\beta}\dot{\alpha}} = 0, \quad R_{\delta}{}^j{}_{\gamma\dot{\gamma}}{}_{\beta\alpha} = -\frac{i}{2}\epsilon_{\delta\dot{\gamma}}\nabla_{\beta}{}^j W_{\alpha\gamma} - \frac{i}{2}\epsilon_{\delta\dot{\gamma}}\nabla_{\alpha}{}^j W_{\beta\gamma} \quad (\text{B.21})$$

- Dimension 2

$$R_{\delta\dot{\delta}}{}_{\gamma\dot{\gamma}}{}_{\beta\alpha} = +\frac{1}{4}\epsilon_{\dot{\delta}\dot{\gamma}}\nabla_{\beta\alpha}W_{\delta\gamma} - \frac{1}{8}\epsilon_{\dot{\delta}\dot{\gamma}}(\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta})\nabla_{\phi\mu}W^{\mu\phi} + \epsilon_{\delta\gamma}\bar{W}_{\dot{\delta}\dot{\gamma}}W_{\beta\alpha} \quad (\text{B.22})$$

$$R_{\delta\dot{\delta}}{}_{\gamma\dot{\gamma}}{}_{\dot{\beta}\dot{\alpha}} = -\frac{1}{4}\epsilon_{\delta\gamma}\bar{\nabla}_{\dot{\beta}\dot{\alpha}}\bar{W}_{\dot{\delta}\dot{\gamma}} + \frac{1}{8}\epsilon_{\delta\gamma}(\epsilon_{\dot{\delta}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\alpha}} + \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\dot{\gamma}\dot{\beta}})\bar{\nabla}_{\dot{\phi}\dot{\mu}}\bar{W}^{\dot{\mu}\dot{\phi}} - \epsilon_{\dot{\delta}\dot{\gamma}}W_{\delta\gamma}\bar{W}_{\dot{\beta}\dot{\alpha}} \quad (\text{B.23})$$

B.3 Dilatation and $U(1)_R$ curvatures

The conformal field strengths for scalings and chiral rotations are

$$R(\mathbb{D}) = dB + 2E^a \wedge F_a - 2E^\alpha \wedge F_\alpha - 2E_{\dot{\alpha}} \wedge F^{\dot{\alpha}} \quad (\text{B.24})$$

$$R(\mathbb{A}) = dA + iE^\alpha \wedge F_\alpha - iE_{\dot{\alpha}} \wedge F^{\dot{\alpha}} \quad (\text{B.25})$$

We group the results by dimension.

- Dimension 1

$$R(\mathbb{D})_{\underline{\beta\alpha}} = R(\mathbb{A})_{\underline{\beta\alpha}} = 0 \quad (\text{B.26})$$

- Dimension 3/2

$$R(\mathbb{D})_{\beta^j \alpha \dot{\alpha}} = +\frac{i}{2} \epsilon_{\beta\alpha} \bar{\nabla}^{\dot{\phi}j} \bar{W}_{\dot{\phi}\dot{\alpha}} \quad (\text{B.27})$$

$$R(\mathbb{D})_{\dot{\beta}^j \alpha \dot{\alpha}} = -\frac{i}{2} \epsilon_{\dot{\beta}\dot{\alpha}} \nabla^{\phi j} W_{\phi\alpha} \quad (\text{B.28})$$

$$R(\mathbb{A})_{\beta^j \alpha \dot{\alpha}} = -\frac{1}{4} \epsilon_{\beta\alpha} \bar{\nabla}^{\dot{\phi}j} \bar{W}_{\dot{\phi}\dot{\alpha}} \quad (\text{B.29})$$

$$R(\mathbb{A})_{\dot{\beta}^j \alpha \dot{\alpha}} = -\frac{1}{4} \epsilon_{\dot{\beta}\dot{\alpha}} \nabla^{\phi j} W_{\phi\alpha} \quad (\text{B.30})$$

- Dimension 2

$$R(\mathbb{D})_{\beta\dot{\beta} \alpha\dot{\alpha}} = \frac{1}{8} \epsilon_{\dot{\beta}\dot{\alpha}} \left(\nabla_\beta^\phi W_{\phi\alpha} + \nabla_\alpha^\phi W_{\phi\beta} \right) - \frac{1}{8} \epsilon_{\beta\alpha} \left(\bar{\nabla}_{\dot{\beta}\dot{\phi}} \bar{W}^{\dot{\phi}}_{\dot{\alpha}} + \bar{\nabla}_{\dot{\alpha}\dot{\phi}} \bar{W}^{\dot{\phi}}_{\dot{\beta}} \right) \quad (\text{B.31})$$

$$R(\mathbb{A})_{\beta\dot{\beta} \alpha\dot{\alpha}} = -\frac{i}{16} \epsilon_{\dot{\beta}\dot{\alpha}} \left(\nabla_\beta^\phi W_{\phi\alpha} + \nabla_\alpha^\phi W_{\phi\beta} \right) - \frac{i}{16} \epsilon_{\beta\alpha} \left(\bar{\nabla}_{\dot{\beta}\dot{\phi}} \bar{W}^{\dot{\phi}}_{\dot{\alpha}} + \bar{\nabla}_{\dot{\alpha}\dot{\phi}} \bar{W}^{\dot{\phi}}_{\dot{\beta}} \right) \quad (\text{B.32})$$

B.4 Isospin curvature

The isospin curvature two-form is

$$R(I)_{ij} = d\Phi_{ij} - \Phi^k_{(i} \wedge \Phi_{j)k} + 4E^\beta_{(i} \wedge F_{\beta j)} - 4E_{\dot{\beta}(i} \wedge F^{\dot{\beta}}_{j)} \quad (\text{B.33})$$

We group the results by dimension.

- Dimension 1

$$R(I)_{\underline{\beta\alpha} ij} = 0 \quad (\text{B.34})$$

- Dimension 3/2

$$R(I)_{\beta^k \alpha \dot{\alpha} ij} = -\frac{i}{2} \epsilon_{\beta\alpha} \delta_j^k \bar{\nabla}^{\dot{\phi}}_i \bar{W}_{\dot{\phi}\dot{\alpha}} - \frac{i}{2} \epsilon_{\beta\alpha} \delta_i^k \bar{\nabla}^{\dot{\phi}}_j \bar{W}_{\dot{\phi}\dot{\alpha}} \quad (\text{B.35})$$

$$R(I)_{\dot{\beta}^k \alpha \dot{\alpha} ij} = +\frac{i}{2} \epsilon_{\dot{\beta}\dot{\alpha}} \delta_j^k \nabla^\phi_i W_{\phi\alpha} + \frac{i}{2} \epsilon_{\dot{\beta}\dot{\alpha}} \delta_i^k \nabla^\phi_j W_{\phi\alpha} \quad (\text{B.36})$$

- Dimension 2

$$R(I)_{\beta\dot{\beta} \alpha\dot{\alpha} ij} = -\frac{1}{4} \epsilon_{\dot{\beta}\dot{\alpha}} \nabla_{ij} W_{\beta\alpha} + \frac{1}{4} \epsilon_{\beta\alpha} \bar{\nabla}_{ij} \bar{W}_{\dot{\beta}\dot{\alpha}} \quad (\text{B.37})$$

B.5 Special conformal curvature

The special conformal curvatures are

$$R(K)^a = dF^a - F^b \wedge \Omega_b^a - F^a \wedge B - 2iF^{\alpha j} \wedge F_{\alpha j}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^a \quad (\text{B.38})$$

$$R(K)^{\alpha i} = dF^{\alpha i} - \frac{1}{2}F^{\alpha i} \wedge B + iF^{\alpha i} \wedge A + F^{\beta i} \wedge \Omega_{\beta}^{\alpha} - F^{\alpha j} \wedge \Phi_j^i + iE_{\dot{\alpha}}^i \wedge F^b \bar{\sigma}_b^{\dot{\alpha}\alpha} \quad (\text{B.39})$$

$$R(K)_{\dot{\alpha} i} = dF_{\dot{\alpha} i} - \frac{1}{2}F_{\dot{\alpha} i} \wedge B - iF_{\dot{\alpha} i} \wedge A + F_{\dot{\beta} i} \wedge \Omega_{\dot{\alpha}}^{\dot{\beta}} + F_{\dot{\alpha} j} \wedge \Phi_j^i + iE^{\alpha}_{\dot{\alpha}} \wedge F_b \sigma_{\alpha\dot{\alpha}}^b \quad (\text{B.40})$$

The components $R(K)_{\underline{\gamma}\underline{\beta}}^A$ are given by

$$R(K)_{\gamma\beta}^{\alpha j} = R(K)^{\dot{\gamma}\dot{\beta}}_{\dot{\alpha} j} = R(K)_{\gamma}^{\dot{\beta} A} = 0 \quad (\text{B.41})$$

$$R(K)_{\gamma\beta\dot{\alpha}}^j = \frac{1}{2}C_{\gamma\beta}\bar{\nabla}_{\dot{\phi}}^j\bar{W}_{\dot{\alpha}}^{\dot{\phi}}, \quad R(K)^{\dot{\gamma}\dot{\beta}}_{\dot{\alpha} j} = -\frac{1}{2}C^{\dot{\gamma}\dot{\beta}}_{\dot{\alpha} j}\nabla^{\phi j}W_{\phi\alpha} \quad (\text{B.42})$$

$$R(K)_{\gamma\beta\alpha\dot{\alpha}} = -C_{\gamma\beta}\nabla_{\alpha\dot{\phi}}\bar{W}_{\dot{\alpha}}^{\dot{\phi}}, \quad R(K)^{\dot{\gamma}\dot{\beta}}_{\alpha\dot{\alpha}} = +C^{\dot{\gamma}\dot{\beta}}_{\alpha\dot{\alpha}}\nabla_{\dot{\alpha}}^{\phi}W_{\phi\alpha} \quad (\text{B.43})$$

The components $R(K)_{\underline{\gamma}^b}^A = -\frac{1}{2}\bar{\sigma}_b^{\dot{\beta}\beta}R(K)_{\underline{\gamma}\beta\dot{\beta}}^A$ are given by

$$R(K)_{\gamma}^j{}_{\beta\dot{\beta}}{}_{\alpha}{}^k = \frac{1}{2}\epsilon_{\gamma\beta}\epsilon^{jk}\nabla_{\alpha\dot{\phi}}\bar{W}_{\dot{\beta}}^{\dot{\phi}}, \quad R(K)_{\dot{\gamma}}^j{}_{\beta\dot{\beta}}{}_{\dot{\alpha}}{}^k = -\frac{1}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\epsilon^{jk}\nabla_{\dot{\alpha}}^{\phi}W_{\phi\beta} \quad (\text{B.44})$$

$$R(K)_{\gamma}^j{}_{\beta\dot{\beta}}{}_{\dot{\alpha}}{}^k = -\frac{i}{4}\epsilon_{\gamma\beta}\bar{\nabla}_{\dot{\beta}}^j\bar{\nabla}_{\dot{\phi}}^k\bar{W}_{\dot{\alpha}}^{\dot{\phi}}, \quad R(K)_{\dot{\gamma}}^j{}_{\beta\dot{\beta}}{}_{\alpha}{}^k = -\frac{i}{4}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla_{\beta}^j\nabla^{\phi k}W_{\phi\alpha} \quad (\text{B.45})$$

$$R(K)_{\gamma}^j{}_{\beta\dot{\beta}}{}_{\alpha\dot{\alpha}} = \frac{i}{2}\epsilon_{\gamma\beta}\bar{\nabla}_{\dot{\beta}}^j\nabla_{\alpha\dot{\phi}}\bar{W}_{\dot{\alpha}}^{\dot{\phi}}, \quad R(K)_{\dot{\gamma}}^j{}_{\beta\dot{\beta}}{}_{\alpha\dot{\alpha}} = \frac{i}{2}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla_{\beta}^j\nabla_{\dot{\alpha}}^{\phi}W_{\phi\alpha} \quad (\text{B.46})$$

The components $R(K)_{cb}^A = \frac{1}{4}\bar{\sigma}_c^{\dot{\gamma}\gamma}\bar{\sigma}_b^{\dot{\beta}\beta}R(K)_{\gamma\dot{\gamma}\beta\dot{\beta}}^A$ are given by

$$R(K)_{\gamma\dot{\gamma}\beta\dot{\beta}}{}^i = \frac{1}{48}\epsilon_{\gamma\dot{\beta}}(\nabla^{ij}\nabla_{\gamma j}W_{\beta\alpha} + \nabla^{ij}\nabla_{\beta j}W_{\gamma\alpha}) \\ + \epsilon_{\gamma\beta}\left(\frac{i}{8}\nabla_{\alpha}^{\dot{\phi}}\bar{\nabla}_{\dot{\gamma}}^j\bar{W}_{\dot{\beta}\dot{\phi}} + \frac{i}{8}\nabla_{\alpha}^{\dot{\phi}}\bar{\nabla}_{\dot{\beta}}^j\bar{W}_{\dot{\gamma}\dot{\phi}} + \frac{1}{4}\bar{W}_{\dot{\gamma}\dot{\beta}}\nabla^{\phi j}W_{\phi\alpha}\right) \quad (\text{B.47})$$

$$R(K)_{\gamma\dot{\gamma}\beta\dot{\beta}}{}_{\dot{\alpha}}{}^i = \frac{1}{48}\epsilon_{\gamma\beta}(\bar{\nabla}_{ij}\bar{\nabla}_{\dot{\gamma}}^j\bar{W}_{\dot{\beta}\dot{\alpha}} + \bar{\nabla}_{ij}\bar{\nabla}_{\dot{\beta}}^j\bar{W}_{\dot{\gamma}\dot{\alpha}}) \\ + \epsilon_{\dot{\gamma}\dot{\beta}}\left(-\frac{i}{8}\nabla_{\dot{\alpha}}^{\phi}\nabla_{\gamma j}W_{\beta\phi} - \frac{i}{8}\nabla_{\dot{\alpha}}^{\phi}\nabla_{\beta j}W_{\gamma\phi} + \frac{1}{4}W_{\gamma\beta}\bar{\nabla}_{\dot{\alpha}}^{\phi}\bar{W}_{\dot{\phi}\dot{\alpha}}\right) \quad (\text{B.48})$$

$$R(K)_{\gamma\dot{\gamma}\beta\dot{\beta}}{}_{\alpha\dot{\alpha}} = -\frac{1}{8}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla_{\dot{\alpha}}^{\phi}\nabla_{\gamma\beta}W_{\phi\dot{\alpha}} - \frac{1}{8}\epsilon_{\gamma\beta}\nabla_{\alpha}^{\dot{\phi}}\bar{\nabla}_{\dot{\gamma}\dot{\beta}}\bar{W}_{\dot{\phi}\dot{\alpha}} \\ + \frac{1}{4}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla_{\gamma}^{\dot{\phi}}(\bar{W}_{\dot{\phi}\dot{\alpha}}W_{\beta\alpha}) + \frac{1}{4}\epsilon_{\dot{\gamma}\dot{\beta}}\nabla_{\beta}^{\dot{\phi}}(\bar{W}_{\dot{\phi}\dot{\alpha}}W_{\gamma\alpha}) \\ + \frac{1}{4}\epsilon_{\gamma\beta}\nabla_{\dot{\gamma}}^{\phi}(W_{\phi\alpha}\bar{W}_{\dot{\beta}\dot{\alpha}}) + \frac{1}{4}\epsilon_{\gamma\beta}\nabla_{\dot{\beta}}^{\phi}(W_{\phi\alpha}\bar{W}_{\dot{\gamma}\dot{\alpha}}) \\ + \frac{i}{8}\epsilon_{\dot{\gamma}\dot{\beta}}(\bar{\nabla}_{\dot{\phi}}^j\bar{W}_{\dot{\phi}\dot{\alpha}})(\nabla_{\gamma}^jW_{\beta\alpha} + \nabla_{\beta}^jW_{\gamma\alpha}) \\ + \frac{i}{8}\epsilon_{\gamma\beta}(\nabla^{\phi j}W_{\phi\alpha})(\bar{\nabla}_{\dot{\gamma}j}\bar{W}_{\dot{\beta}\dot{\alpha}} + \bar{\nabla}_{\dot{\beta}j}\bar{W}_{\dot{\gamma}\dot{\alpha}}) \quad (\text{B.49})$$

In the expression for $R(K)_{cba}$ we have reordered the derivatives given in the original expression (2.44); where we originally had $\nabla_{\gamma\beta}\nabla_{\dot{\alpha}}^{\phi}W_{\phi\dot{\alpha}}$, we have applied the commutation relations to give $\nabla_{\dot{\alpha}}^{\phi}\nabla_{\gamma\beta}W_{\phi\dot{\alpha}}$ (and similarly for the complex conjugate). This simplifies the analysis since the bosonic derivative is now furthest to the left.

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